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Existence, uniqueness and multiplicity of positive solutions for Schrödinger–Poisson system with singularity ☆

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A R T I C L E   I N F O

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A B S T R A C T

In this paper, we consider the following Schrödinger–Poisson system with singularity

$$\begin{cases} -\Delta u + \eta\phi u = \mu u^{-r}, & \text{in } \Omega, \\ -\Delta\phi = u^2, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain with boundary $\partial\Omega$, $\eta = \pm 1$, $r \in (0,1)$ is a constant, $\mu > 0$ is a parameter. We obtain the existence and uniqueness of positive solution for $\eta = 1$ and any $\mu > 0$ by using the variational method. The existence and multiplicity of solutions for the system are also considered for $\eta = -1$ and $\mu > 0$ small enough by using the method of Nehari manifold.

1. Introduction

In this paper, we consider the following singular Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \eta\phi u = \mu u^{-r}, & \text{in } \Omega, \\ -\Delta\phi = u^2, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain with boundary $\partial\Omega$, $\eta = \pm 1$, $r \in (0, 1)$ is a constant, $\mu > 0$ is a parameter.

This problem is derived from the recent research on the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + q\phi f(u) = g(x, u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = 2F(u), & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

Recently, the existence, nonexistence, multiplicity results, ground state and sign-changing solutions of system (1.2) have been studied widely by using the modern variational method and critical point theory under various assumptions of nonlocal term f and nonlinear term g, see [9,11,12,22,16,1,6,5,30,19,18,28,31,20], etc.

There are also many references which investigated Schrödinger–Poisson system in bounded domain, see [7,3,4]. In [3], the following system involving the critical growing nonlocal term was considered

$$\begin{cases} -\Delta u = \lambda u + q|u|^3 u\phi, & \text{in } B_R, \\ -\Delta\phi = q|u|^5, & \text{in } B_R, \\ u = \phi = 0, & \text{on } \partial B_R, \end{cases}$$

where B_R is a ball in $\mathbb{R}^3$ centered at the origin and with radius R. The existence and nonexistence results were obtained by discussing the scope of the parameter λ. By using the methods of a cut-off function and the variational arguments, in [4], the authors studied the following Schrödinger–Poisson system in bounded domain

$$\begin{cases} -\Delta u + \varepsilon q\phi f(u) = \eta|u|^{p-1}u, & \text{in } \Omega, \\ -\Delta\phi = 2qF(u), & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $p \in (1, 5)$, $q > 0$, ε, $\eta = \pm 1$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $F(t) = \int_0^t f(s)ds$. They obtained the existence and multiplicity results assuming on f a subcritical growth condition and they also considered the existence and nonexistence results under the critical case.

The following singular semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda u^p + \mu u^{-r}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geqslant 3$, $p > 0$, $r \in (0, 1)$, has been extensively studied, see [17,25,24,29,15,8,26,2,27], etc. When $\lambda \equiv 0$, the existence of solutions has been studied in [10,13,17]. When $p \in (1, 2^* - 1)$, the existence and multiplicity of solutions for (1.3) have been studied in [25,29,15,2,27] for all $\mu > 0$ and $\lambda > 0$ small enough. The existence of multiple solutions of (1.3) for $p = 2^* - 1$ and $\lambda > 0$ small enough has been considered in [2]. In [24], the existence result for $p \in (0, 1)$ was considered.

Recently, in [21], the following singular Kirchhoff type problem which possesses the nonlocal term $(b \int_\Omega |\nabla u|)\Delta u$ has been considered

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2)\Delta u = \lambda u^3 + \mu u^{-r}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$