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Essential pseudospectra and essential norms of band-dominated operators

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ABSTRACT

An operator A on an l^p-space is called band-dominated if it can be approximated, in the operator norm, by operators with a banded matrix representation. The coset of A in the Calkin algebra determines, for example, the Fredholmness of A, the Fredholm index, the essential spectrum, the essential norm and the so-called essential pseudospectrum of A. This coset can be identified with the collection of all so-called limit operators of A. It is known that this identification preserves invertibility (hence spectra). We now show that it also preserves norms and in particular resolvent norms (hence pseudospectra). In fact we work with a generalization of the ideal of compact operators, so-called $\mathcal{P}$-compact operators, allowing for a more flexible framework that naturally extends to l^p-spaces with $p \in \{1, \infty\}$ and/or vector-valued l^p-spaces.

1. Introduction

This first section comes as a rough guide to this paper. Proper definitions and theorems are given in later sections.

We study bounded linear operators on a Banach space $\mathbf{X}$. Most of the time, $\mathbf{X}$ is an l^p sequence space with $1 \le p \le \infty$, index set $\mathbb{Z}^N$ and values in another Banach space X, so that an operator on $\mathbf{X} = l^p(\mathbb{Z}^N, X)$ can be identified, in a natural way, with an infinite matrix (a_{ij}) with indices $i, j \in \mathbb{Z}^N$ and all a_{ij} being operators $X \to X$.

For such an operator A on $\mathbf{X}$, write $A \in \mathcal{K}_0(\mathbf{X}, \mathcal{P})$ if its matrix (a_{ij}) has finite support (i.e. only finitely many nonzero entries), and write $A \in \mathcal{A}_0(\mathbf{X})$ if its matrix is a band matrix (i.e. it has only finitely many nonzero diagonals). Clearly, $\mathcal{A}_0(\mathbf{X})$ is an algebra containing $\mathcal{K}_0(\mathbf{X}, \mathcal{P})$ as a (two-sided) ideal. Denote the

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closure, in the $\mathbf{X} \to \mathbf{X}$ operator norm, of $\mathcal{A}_0(\mathbf{X})$ by $\mathcal{A}(\mathbf{X})$ and the closure of $\mathcal{K}_0(\mathbf{X}, \mathcal{P})$ by $\mathcal{K}(\mathbf{X}, \mathcal{P})$. Then $\mathcal{A}(\mathbf{X})$ is a Banach algebra containing $\mathcal{K}(\mathbf{X}, \mathcal{P})$ as a closed ideal.[1]

Operators in $\mathcal{A}(\mathbf{X})$ are called band-dominated operators. The ideal $\mathcal{K}(\mathbf{X}, \mathcal{P})$ is a generalization of the set of compact operators: If $\dim X < \infty$ then $\mathcal{K}(\mathbf{X}, \mathcal{P})$ coincides with the set $\mathcal{K}(\mathbf{X})$ of all compact operators on $\mathbf{X}$ (except in the somewhat pathological cases $p = 1$ and $p = \infty$); otherwise it does not – as already $\mathcal{K}_0(\mathbf{X}, \mathcal{P})$ contains non-compact operators. Recall that $\mathcal{K}(\mathbf{X})$ is a closed ideal in the algebra $\mathcal{L}(\mathbf{X})$ of all bounded linear operators $\mathbf{X} \to \mathbf{X}$.

For $A \in \mathcal{A}(\mathbf{X})$, the coset

$$A + \mathcal{K}(\mathbf{X}, \mathcal{P}) \text{ in the quotient algebra } \mathcal{A}(\mathbf{X})/\mathcal{K}(\mathbf{X}, \mathcal{P}) \tag{1.1}$$

is of interest. If $\mathcal{K}(\mathbf{X}, \mathcal{P}) = \mathcal{K}(\mathbf{X})$ then the quotient norm of (1.1) is the usual essential norm of A, the spectrum of (1.1) is the essential spectrum of A, and the invertibility of (1.1) corresponds to A being a Fredholm operator (i.e. having a finite-dimensional kernel and a finite-codimensional range). In the general case one gets generalized versions of these quantities and properties.

In either case, the coset (1.1) is an interesting but complicated object. Our strategy for its study is a localization technique that replaces this one complicated object by a family of many simpler objects. The key observation is that, by the definition of the ideal $\mathcal{K}(\mathbf{X}, \mathcal{P})$, the coset (1.1) depends only (and exactly) on the asymptotic behavior of the matrix behind A. This asymptotic behavior is extracted as follows: For every $k \in \mathbb{Z}^N$, let $V_k : \mathbf{X} \to \mathbf{X}$ denote the k-shift operator that maps $(x_i)_{i\in\mathbb{Z}^N}$ to $(y_i)_{i\in\mathbb{Z}^N}$ with $y_{i+k} = x_i$, and then look at the translates $V_{-k}AV_k$ of A. The simpler objects that characterize the coset (1.1) are the partial limits of the family $(V_{-k}AV_k)_{k\in\mathbb{Z}^N}$ of all translates of A with respect to the so-called $\mathcal{P}$-topology, to be described below, that corresponds to entry-wise norm convergence of the matrix. More precisely, if $h = (h_1, h_2, \ldots)$ is a sequence in $\mathbb{Z}^N$ with $|h_n| \to \infty$ and $V_{-h_n}AV_{h_n}$ converges in that topology then we denote the limit by A_h and call it the limit operator of A with respect to the sequence h. Doing this with all such sequences that produce a limit operator yields the collection

$$\sigma_{\text{op}}(A) := \{A_h \,:\, h = (h_1, h_2, \ldots),\, h_n \in \mathbb{Z}^N,\, |h_n| \to \infty,\, A_h := \mathcal{P}\text{-}\lim V_{-h_n}AV_{h_n} \text{ exists}\} \tag{1.2}$$

of all limit operators – the so-called operator spectrum of A. We have used sequences h to address our partial limits of $(V_{-k}AV_k)_{k\in\mathbb{Z}^N}$. The same set (1.2) can also be constructed as follows [31,40]: Extend the mapping $\varphi_A : k \in \mathbb{Z}^N \mapsto V_{-k}AV_k \in \mathcal{A}(\mathbf{X})$ $\mathcal{P}$-continuously to the (Stone–Čech) boundary $\partial\mathbb{Z}^N$ of $\mathbb{Z}^N$. Then (1.2) exactly collects the values of φ_A on $\partial\mathbb{Z}^N$. Enumerating the set (1.2) via $\partial\mathbb{Z}^N$ (rather than via the set of all sequences h in $\mathbb{Z}^N$ for which A_h exists) has the benefit that the index set $\partial\mathbb{Z}^N$ is independent of A, so that two instances of (1.2) can be added or multiplied elementwise. Under these operations, the map $A \mapsto \varphi_A|_{\partial\mathbb{Z}^N} =$ (1.2) turns out to be an algebra homomorphism. Now the crucial point is that $\mathcal{K}(\mathbf{X}, \mathcal{P})$ is exactly the kernel of that homomorphism $A \mapsto$ (1.2), whence (1.1) $\mapsto$ (1.2) is a well-defined algebra isomorphism.[2] In short: The set (1.2) nicely reflects the coset (1.1). Actually, besides $A \in \mathcal{A}(\mathbf{X})$, there is one technical condition to make this identification between the coset (1.1) and the set (1.2) work: To make sure that (1.2) is large enough, we have to assume that $\{V_{-k}AV_k : k \in \mathbb{Z}^N\}$ has a sequential compactness property, namely that every sequence h in $\mathbb{Z}^N$ with $|h_n| \to \infty$ has a subsequence g for which the $\mathcal{P}$-limit A_g exists, in which case we call A a rich operator (in the sense that (1.2) is rich enough to reflect all[3] of (1.1)).

[1] We will explain the notation $\mathcal{K}(\mathbf{X}, \mathcal{P})$ later and say what $\mathcal{P}$ is.

[2] To oversimplify matters, think of continuous functions f on a compact set D. Then the subspace (actually the ideal) $C_0(D)$ of continuous functions with zero boundary values is the kernel of the algebra homomorphism $f \mapsto f|_{\partial D}$, whence the coset of f modulo $C_0(D)$ can be identified with $f|_{\partial D}$, by the fundamental homomorphism theorem.

[3] In fact, the map $A \mapsto \sigma_{\text{op}}(A) =$ (1.2) sends some operators $A \in \mathcal{A}(\mathbf{X})$ to $\varnothing$. For some other $A \in \mathcal{A}(\mathbf{X})$, limit operators exist in one "direction" but not in another. Some of the latter A are not in $\mathcal{K}(\mathbf{X}, \mathcal{P})$ but have $\sigma_{\text{op}}(A) = \{0\}$, such as our first example in Remark 3.6. These problems are eliminated by imposing existence of sufficiently many limit operators, i.e. richness of A.