J. Math. Anal. Appl. 437 (2016) 310–329

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Energy concentration of the focusing energy-critical fNLS

Yonggeun Cho [a,*], Gyeongha Hwang [b], Yong-Sun Shim [c]

[a] *Department of Mathematics, and Institute of Pure and Applied Mathematics, Chonbuk National University, Jeonju 561-756, Republic of Korea*
[b] *National Center for Theoretical Sciences, Mathematics Division, National Taiwan University, Taipei 10617, Taiwan*
[c] *Department of Mathematics, POSTECH, Pohang 790-784, Republic of Korea*

ARTICLE INFO

Article history:
Received 7 October 2015
Available online 7 January 2016
Submitted by K. Nishihara

Keywords:
Global well-posedness
Blowup
Energy concentration
Focusing energy-critical fNLS
Concentration compactness
Energy trapping

ABSTRACT

We consider the fractional nonlinear Schrödinger equation (fNLS) with non-local dispersion $|\nabla|^\alpha$ and focusing energy-critical Hartree type nonlinearity $[-(|x|^{-2\alpha} * |u|^2)u]$. We consider the energy-concentration phenomena of radial blowup solutions near the maximal existence time. We use the concentration-compactness approach of [19] for confined case and kinetic energy trapping approach of [21] for unconfined case.

1. Introduction

The fractional nonlinear Schrödinger equation is a generalization of NLS in the aspect of fractional quantum mechanics [20]. Namely, the associated equation for the wave function results in the fractional Schrödinger equations, which contains a nonlocal fractional derivative operator $|\nabla|^\alpha = (-\Delta)^{\frac{\alpha}{2}}$. We list basic notations in section 2. In this paper we consider the Cauchy problem of the focusing fractional nonlinear Schrödinger equations:

$$\begin{cases} i\partial_t u = |\nabla|^\alpha u - V(u)u \ \text{ in } \ \mathbb{R}^{1+d}, \ \ d \geq 3, \\ u(t_0, x) = \varphi(x) \ \text{ in } \ \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $V(u) = (|x|^{-2\alpha} * |u|^2)$ and $1 < \alpha < \min(2, \frac{d}{2})$. The potential $(-V(u))$ shows the way of focusing interaction of particles and it is an archetypal model for blowup of solution. The equation (1.1) is of

* Corresponding author.
E-mail addresses: changocho@jbnu.ac.kr (Y. Cho), ghhwang@ncts.ntu.edu.tw (G. Hwang), shim@postech.ac.kr (Y.-S. Shim).

http://dx.doi.org/10.1016/j.jmaa.2015.12.060

$\dot{H}^{\frac{\alpha}{2}}$-scaling invariance (so-called energy-critical). That is, if u is a solution of (1.1), then for any $\lambda > 0$ the scaled function u_λ, given by

$$u_\lambda(t,x) = \lambda^{\frac{d}{2}-\frac{\alpha}{2}} u(\lambda^\alpha t, \lambda x),$$

is also a solution to (1.1). So we consider the solutions $u \in C(I_*; \dot{H}^{\frac{\alpha}{2}})$ in this paper. The interval I_* denotes the maximal existence time and $I_* = (-T_*, T^*)$ for $T_*, T^* \in (0, \infty]$. Due to the critical nonlinearity the time of existence no longer depends on the $\dot{H}^{\frac{\alpha}{2}}$ norm of initial data. Instead it relies on the profiles of the data. Hence the situation become more subtle. Unlike NLS the non-locality of $|\nabla|^\alpha$ gives a significant obstacle to the analysis for well-posedness and blowup, and what is worse its weak dispersiveness leads us to the lack of or regularity loss of linear estimates such as time decay, Strichartz estimate and bilinear estimate. Nevertheless, there have been a lot of effort to overcome the difficulties and results begin to emerge. For instance see [13,9,4,5,14,16,2,7].

As a well-known fact, a solution readily satisfies the energy conservation law: for all $t \in I_*$

$$E(u(t)) := \mathcal{K}(u(t)) \text{ [Kinetic energy]} + \mathcal{P}(u(t)) \text{ [Potential energy]} = E(\varphi), \tag{1.2}$$

where

$$\mathcal{K}(u) = \frac{1}{2}\int ||\nabla|^{\frac{\alpha}{2}} u(x)|^2\, dx, \quad \mathcal{P}(u) = -\frac{1}{4}\int V(u)|u|^2 dx.$$

Also if $u \in C(I_*; L^2)$, then it satisfies the mass conservation

$$m(u(t)) := \|u(t)\|_{L^2}^2 = m(\varphi).$$

Definition 1.1. We say the kinetic energy is confined on I if $\sup_{t\in I} \mathcal{K}(u(t)) < \infty$ and unconfined on I if $\sup_{t\in I} \mathcal{K}(u(t)) = \infty$.

In this paper we aim to demonstrate the concentration phenomenon of kinetic energy for blowup solutions to (1.1). We consider radial solutions to get around the difficulties described as above, which make it especially hard to find concentration points of solution. To expedite matters we define the blowup of solution by following the notion of the antecedent results [3,11,18].

Definition 1.2. It is said that u blows up backward in time if $\|u\|_{S_\alpha((-T_*,0])} = +\infty$ and forward in time if $\|u\|_{S_\alpha([0,T^*))} = +\infty$. Hereafter $S_\alpha(I)$ denotes $L_I^6 L_x^{2d/(d-4\alpha/3)}$ for an interval I.

The space S_α was used to study the well-posedness in [7]. The blowup may occur at time infinity but it occurs at finite time under some conditions. The associated finite time blowup can be stated as follows:

Blowup. *(See Theorem 4.1 of [7].) Let $d \geq 3$ and $1 < \alpha < \min(2, \frac{d}{2})$. Suppose that $u \in C(I_*; H^{\frac{\alpha}{2}}_{rad})$ is a solution to (1.1) with φ satisfying one of the following*

$$E(\varphi) < 0,$$
$$0 \leq E(\varphi) < E(W_\alpha) \ \text{and}\ \||\nabla|^{\frac{\alpha}{2}}\varphi\|_{L^2} \geq \||\nabla|^{\frac{\alpha}{2}} W_\alpha\|_{L^2}.$$

Then $T_*, T^* < \infty$.

W_α is the ground state of (1.1) which is a unique positive radial solution of the elliptic equation $|\nabla|^\alpha W = V(W)W$. In [12] the authors showed that any solution of the elliptic equation is a constant multiple, dilation