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# An inner–outer factorization in $\ell^p$ with applications to ARMA processes

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ABSTRACT

The following inner–outer type factorization is obtained for the sequence space $\ell^p$: if the complex sequence $\mathbf{F} = (F_0, F_1, F_2, \ldots)$ decays geometrically, then for any $p$ sufficiently close to 2 there exist $\mathbf{J}$ and $\mathbf{G}$ in $\ell^p$ such that $\mathbf{F} = \mathbf{J} * \mathbf{G}$; $\mathbf{J}$ is orthogonal in the Birkhoff–James sense to all of its forward shifts $S\mathbf{J}, S^2\mathbf{J}, S^3\mathbf{J}, \ldots$; $\mathbf{J}$ and $\mathbf{F}$ generate the same $S$-invariant subspace of $\ell^p$; and $\mathbf{G}$ is a cyclic vector for $S$ on $\ell^p$. These ideas are used to show that an ARMA equation with characteristic roots inside and outside of the unit circle has Symmetric-$\alpha$-Stable solutions, in which the process and the given white noise are expressed as causal moving averages of a related i.i.d. S$\alpha$S white noise. An autoregressive representation of the process is similarly obtained.



## 1. Introduction

In this paper we begin to examine, under certain circumstances, a possible "inner–outer" factorization for the class $\ell^p_A$ of analytic functions $f$ on the open unit disk $\mathbb{D}$ whose Taylor coefficients belong to the classical sequence space $\ell^p$. When $p = 2$, the sequence space $\ell^2_A$ is the classical Hardy space $H^2$, and a theorem of Beurling says that every $f \in \ell^2_A$ can be factored, uniquely up to unimodular constants, as

$$f = JG, \tag{1.1}$$

where $J$ is an inner function and $G$ is an outer function. More specifically, the function $J$ can be obtained by the formula $J = f - \widehat{f}$, where $\widehat{f}$ is the orthogonal projection of $f$ onto $S[f]$, equivalently

$$J \perp S^k J, \quad k \geq 1. \tag{1.2}$$

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In the above, $S$ is the unilateral shift operator $Sg = zg$ on $\ell^2_A$ and $[f]$ is the $S$-invariant subspace generated by $f$. Furthermore, the function $G$ satisfies $[G] = \ell^2_A$, i.e., $G$ is a cyclic vector for $S$, and the function $J$ satisfies $[J] = [f]$. The classic text [13] provides a full account of all this.

When $p \neq 2$, the invariant subspaces of $\ell^p_A$ can be quite complicated and, especially when $p > 2$, can have many dramatic pathological properties [1]. In short, the $S$-invariant subspaces for $p \neq 2$ are not understood much at all. Even when $p = 1$, where $\ell^1_A$ is an algebra of continuous functions on the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$ (known as the Wiener algebra), the $S$-invariant subspaces, which turn out to be the ideals of $\ell^1_A$, still have delicate, but pathological, properties [18].

In this paper, we obtain a partial Beurling-type factorization as in (1.1) by replacing the standard Hilbert space orthogonality in $\ell^2_A$, used in (1.2) to define the inner factor $J$, by the Birkhoff–James orthogonality relationship $\perp_p$ in $\ell^p_A$ (see (3.2) below). Our main theorem is the following.

**Theorem 1.3.** *If $f$ is analytic in a neighborhood of $\overline{\mathbb{D}}$, then for any $p$ sufficiently close to 2, there exist functions $G$ and $J$, analytic in a neighborhood of $\overline{\mathbb{D}}$, such that*

$$f = JG$$

*where $G$ is cyclic vector for $S$ on $\ell^p_A$ and $J$ satisfies*

$$J \perp_p S^k J \quad k = 1, 2, 3, \ldots.$$

*Furthermore, we have $[f] = [J]$ in $\ell^p_A$. The functions $J$ and $G$ are unique up to multiplicative constants.*

As with Beurling's Theorem, notice that we have $J \perp_p S^k J$ for all $k \geq 1$, which is an equivalent to $J$ being inner when $p = 2$, and $[G] = \ell^p_A$, which is equivalent to $G$ being outer when $p = 2$. One might expect that "inner" in this situation depends on $p$. Indeed, the "inner" factor $J$ when $p = 2$ is actually an inner function in the classical sense (i.e., a bounded analytic function on $\mathbb{D}$ with unimodular radial boundary values almost everywhere on the circle $\mathbb{T}$), up to a constant factor. When $p \neq 2$, one can have the condition $J \perp_p S^k J$ for all $k \geq 1$ but without $J$ being inner in the classical sense (see the examples at the ends of Sections 3 and 5, respectively).

The next section sets forth the notation used in this paper and reviews the related function theory. Section 3 contains the development of the main theorem. The analytical tools derived in Section 3 are applied in Section 5 to solve a problem concerning Autoregressive Moving Average (ARMA) processes. It is shown that an infinite-variance ARMA model has a causal stationary solution, even if its characteristic polynomials have roots both inside and outside the unit circle.

The modest $\ell^p$ factorization given in Theorem 1.3 is sufficient to solve the intended application to ARMA processes. However, one can see that there is much work to be done to obtain, if possible, a general "inner–outer" factorization for all $f \in \ell^p_A$, not necessarily smooth up to the boundary. We invite the reader to join in the discussion.

## 2. Preliminaries

For $1 \leq p < \infty$ define $\ell^p$ to be the set of sequences

$$\mathbf{a} = (a_0, a_1, \ldots)$$

of complex numbers for which

$$\|\mathbf{a}\|_p := \left( \sum_{k=0}^{\infty} |a_k|^p \right)^{1/p} < \infty.$$