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A gradient estimate to a degenerate parabolic equation with a singular absorption term: The global quenching phenomena

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ABSTRACT

We prove global existence of nonnegative solutions to the one dimensional degenerate parabolic problems containing a singular term. We also show the global quenching phenomena for L^1 initial datums. Moreover, the free boundary problem is considered in this paper.

Contents

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1. Introduction

This paper deals with the one dimensional degenerate parabolic equation on a given open bounded interval $I = (L_1, L_2)$

$$\begin{cases} \partial_t u - (|u_x|^{p-2}u_x)_x + \chi_{\{u>0\}}u^{-\beta} = 0 & \text{in } I \times (0,\infty), \\ u(L_1,t) = u(L_2,t) = 0 & t \in (0,\infty), \\ u(x,0) = u_0(x) & \text{in } I, \end{cases} \tag{1}$$

where $\beta \in (0,1)$, $p > 2$, $u_0 \geq 0$ and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points (x,t) where $u(x,t) > 0$. The absorption term $\chi_{\{u>0\}}u^{-\beta}$ becomes singular when u is near to 0 (but note that we are imposing $\chi_{\{u>0\}}u^{-\beta} = 0$ if $u = 0$). We shall also consider the associated Cauchy problem (formally equivalent (1) when $I = \mathbb{R}$).

Problem (1) can be considered as a limit model of a class of problems arising in Chemical Engineering corresponding to catalyst kinetics of Langmuir–Hinshelwood type (see, e.g. [20, p. 68]). Here we assume that the diffusion coefficient, $D = |u_x|^{p-2}$, depends on the gradient of the concentration. From a mathematical point of view, the pioneering papers on this class of models were due to Phillips [18] and Bandle and Brauner [1], for the case $p = 2$ (even posed on an open bounded set Ω of $\mathbb{R}^N$). Besides, other authors also considered the semilinear case ($p = 2$); see, e.g. [16,6,21,8,5] and their references. The case of quasilinear diffusion operators was already considered in [13] (for a different diffusion term). We also mention here the case of the quasilinear problem of porous medium type studied in [14]. Recently, problem (1) was analyzed in the paper [11] (even under a more general formulation, see also the study of the associated stationary problem [17]) but the proof of the existence of a weak solution (as limit of solutions of approximate non-singular problems) is not completely well justified. One of the main goals of this paper is to get some sharper a priori estimates on the (spatial) gradient of the approximate solutions to pass to the limit in the approximation of the singular term of the equation.

Roughly speaking, the a priori gradient estimate that we shall prove is of the type

$$|\partial_x u(x,t)| \leq C u^{1-\frac{1}{\gamma}}(x,t), \quad \text{for a.e. } (x,t) \in I \times (0,\infty), \tag{2}$$

for a suitable constant $C > 0$, and the exponent

$$\gamma = \frac{p}{p+\beta-1}. \tag{3}$$

Estimates of this type were already obtained (for the case of $p = 2$ and bounded initial data) in [18,6,21]. The degeneracy of the diffusion operator when $p > 2$ leads, obviously, to a considerable amount of additional technical difficulties (see, e.g. the study of the unperturbed equation made in [12]). In addition, as in [5], we want to consider also the case of possibly unbounded initial data. Let us mention that the exponent γ given by (3) plays a fundamental role. It arises, in a natural way, when considering the associate stationary problem. It is not difficult to show that in that case the estimate (2) becomes an equality, for a suitable constant C. This is the reason why some authors call this type of gradient estimates as “sharp gradient estimates” (see, e.g., [2] for a general exposition of this type of estimates).

As mentioned before, a very delicate point is to require a suitable integrability to the singular term of the equation. So, before stating our main results, let us define the notion of weak solution of equation (1) which we shall consider in this paper.

Definition 1. Given $0 \leq u_0 \in L^1(I)$, a function u is called a weak solution of (1) if $u \in L^p_{loc}(0,\infty; W_0^{1,p}(I)) \cap L^\infty_{loc}(\overline{I} \times (0,\infty)) \cap \mathcal{C}([0,\infty); L^1(I))$, $u^{-\beta}\chi_{\{u>0\}} \in L^1(I \times (0,\infty))$, and u satisfies equation (1) in the sense of distributions $\mathcal{D}'(I \times (0,\infty))$, i.e.: