J. Math. Anal. Appl. 437 (2016) 474–484

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A regularity criterion for the Navier–Stokes equations based on the gradient of one velocity component

Zdeněk Skalák

Institute of Hydrodynamics, Academy of Sciences of the Czech Republic, Pod Paťankou 30/5, 166 12 Prague 6, Czech Republic

A R T I C L E I N F O

Article history:
Received 4 October 2015
Available online 13 January 2016
Submitted by P.G. Lemarie-Rieusset

Keywords:
Navier–Stokes equations
Regularity of solutions
Regularity criteria

A B S T R A C T

We show that if u is a Leray solution to the Navier–Stokes equations in the full three-dimensional space with an initial condition from $W^{1,2}_{0,\sigma}$, $T > 0$ and $u \in L^t(0,T;L^s)$, where $2/t + 3/s = 59/30$ for $s \in (2, 30/13]$ and $2/t + 3/s = 7/4 + 1/(2s)$ for $s \in (30/13, 3)$ then u is regular on $(0,T)$. We prove our result as a special case of a more general method which might possibly bring a further improvement.

1. Introduction

We consider the Navier–Stokes equations in the full three-dimensional space, i.e.

$$\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0 \quad \text{in } \mathbf{R}^3 \times (0,\infty), \tag{1}$$

$$\nabla \cdot u = 0 \quad \text{in } \mathbf{R}^3 \times (0,\infty), \tag{2}$$

$$u|_{t=0} = u_0, \tag{3}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p = p(x,t)$ denote the unknown velocity and pressure and $u_0 = u_0(x) = (u_{01}(x), u_{02}(x), u_{03}(x))$ is a given initial velocity.

It is known that for $u_0 \in L^2_\sigma$ (solenoidal functions from L^2) the problem (1)–(3) possesses at least one global weak solution u satisfying the energy inequality $||u(t)||_2^2/2 + \int_0^t ||\nabla u(\tau)||_2^2 \, d\tau \le ||u_0||_2^2/2$ for every $t \ge 0$ (see [13] or [23]). Such solutions are called Leray solutions. If $u_0 \in W^{1,2}_\sigma$ (solenoidal functions from the standard Sobolev space $W^{1,2}$) then Leray solutions are regular on some (possibly small) time interval. It is a classical question to ask whether or not these solutions are regular on an arbitrary interval $(0,T)$, $T > 0$, i.e. whether or not $\nabla u \in L^\infty_{loc}([0,T); L^2)$, $u \in L^2_{loc}(0,T; W^{2,2})$ and (subsequently) $u \in C^\infty_{loc}((0,T) \times R^3)$ (see

E-mail address: skalak@mat.fsv.cvut.cz.

http://dx.doi.org/10.1016/j.jmaa.2016.01.023
0022-247X/

[23]). This renowned problem has not yet been solved and seems to be beyond the scope of the present techniques. Nevertheless, there exist many criteria in the literature ensuring the positive answer – see for example [1,2,4,3,8,10,9,11,14–19,28] and the literature cited there. Mention here specifically the following classical result from [1]: a Leray solution u with the initial condition from $W^{1,2}_\sigma$ is regular on $(0,T)$ if $\nabla u \in L^t(0,T;L^s)$, where $2/t + 3/s = 2$ and $s \in (3/2,\infty)$. Notice that due to the last equality with the number two on its right hand side, this result is optimal from the scaling point of view. In this paper we are interested in criteria of Serrin-type based on the gradient of only one velocity component, ∇u_3. These criteria have been studied in the last fifteen years, see [12,20,29,30,21,22,27,26]. Their investigation seems to be more difficult than the investigation of akin criteria based on one partial derivative of the velocity field, $\partial_3 u$, where some optimal results were reached several years ago by the use of relatively elementary techniques (see [2] and [12]). In the case of criteria based on ∇u_3 such optimal results were proved only recently, see [5,6,25] and the methods used in these papers are by no means elementary. The following list shows the present state of the art: if $T > 0$, then the Leray solution u with the initial condition from $W^{1,2}_\sigma$ is regular on $(0,T)$ if $\nabla u_3 \in L^t(0,T;L^s)$, where

$$\frac{2}{t} + \frac{3}{s} \le 2, \quad s \in \Big(\frac{3}{2}, \frac{9}{5}\Big], \quad \text{see [6]} \tag{4}$$

$$\frac{2}{t} + \frac{3}{s} \le 2, \quad s \in \Big(\frac{9}{5}, 2\Big), \quad \text{see [5]} \tag{5}$$

$$\frac{2}{t} + \frac{3}{s} \le 2, \quad s = 2, \quad \text{see [25]} \tag{6}$$

$$\frac{2}{t} + \frac{3}{s} \le \frac{15s+1}{8s}, \quad s \in (2,3), \quad \text{see [26]} \tag{7}$$

$$\frac{2}{t} + \frac{3}{s} \le \frac{7}{4} + \frac{1}{2s}, \quad s \in \Big[3, \frac{10}{3}\Big), \quad \text{see [29]} \tag{8}$$

$$\frac{2}{t} + \frac{3}{s} \le \frac{7}{4} + \frac{1}{2s}, \quad s \in \Big[\frac{10}{3}, \infty\Big), \quad \text{see [21].} \tag{9}$$

Remark 1. In fact in [5] the authors proved the following result: if moreover the vorticity $\nabla \times u_0 \in L^{3/2}$ then Leray solutions satisfying $u_3 \in L^q(0,T;\dot{H}^{1/2+2/q})$, $q \in (4,6)$, are regular on $(0,T)$. It is obvious that (5) follows as a direct consequence: Namely, if $\nabla u_3 \in L^q(0,T;L^p)$, where $2/q + 3/p = 2$ and $p \in (9/5,2)$ and $q \in (4,6)$, then $\nabla u_3 \in L^q(0,T;\dot{H}^{2/q-1/2})$ and $u_3 \in L^q(0,T;\dot{H}^{2/q+1/2})$. Applying the criterion from [5] gives the regularity of u. The criterion from [6] is the extension of the result from [5] for $q \in (4,\infty)$ and it implies immediately (4).

In this paper we will focus on (7) and we will prove the following theorem:

Theorem 1. *Let u be a global weak solution to (1)–(3) corresponding to the initial condition $u_0 \in W^{1,2}_\sigma$ and satisfying the energy inequality. Let $T > 0$, $\nabla u_3 \in L^t(0,T;L^s)$ and*

$$\frac{2}{t} + \frac{3}{s} \le \frac{59}{30}, \quad s \in \Big(2, \frac{30}{13}\Big], \tag{10}$$

$$\frac{2}{t} + \frac{3}{s} \le \frac{7}{4} + \frac{1}{2s}, \quad s \in \Big(\frac{30}{13}, 3\Big). \tag{11}$$

Then u is regular on $(0,T)$.

The criterion from Theorem 1 improves (7), but it is not optimal. As a second goal of this paper we will present the method of the proof of Theorem 1 as a special case of a more general method. This method suitably combines the ideas from [29] and Lemma 1. We will show that Theorem 1 can be proved using