J. Math. Anal. Appl. 437 (2016) 498–512

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Norm estimations for the Moore–Penrose inverse of multiplicative perturbations of matrices

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ARTICLE INFO

Article history:
Received 19 August 2015
Available online 14 January 2016
Submitted by J.A. Ball

Keywords:
Moore–Penrose inverse
Multiplicative perturbation
Norm upper bound

ABSTRACT

Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by $B = D_1^* A D_2$, where $D_1 \in \mathbb{C}^{m\times m}$ and $D_2 \in \mathbb{C}^{n\times n}$ are both nonsingular. New norm upper bounds for $\|B^\dagger - A^\dagger\|_F$ and $\|B^\dagger - A^\dagger\|_2$ are derived, where $A^\dagger$ and $B^\dagger$ are the Moore–Penrose inverses of A and B, respectively. The main results of Meng and Zheng (2015) [10] are improved in the cases of the Frobenius norm and the spectral norm.

1. Introduction

Throughout this paper, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{C}$, $\mathbb{C}^{m\times n}$ are the sets of positive integers, real numbers, nonnegative real numbers, complex numbers and $m \times n$ complex matrices, respectively. For any $A \in \mathbb{C}^{m\times n}$, let $\mathcal{R}(A)$, $\mathcal{N}(A)$, A^*, $\|A\|_2$, $\|A\|_F$ and $A^\dagger$ denote the range, the null space, the conjugate transpose, the spectral norm, the Frobenius norm and the Moore–Penrose inverse of A [15], respectively, where $A^\dagger$ is the unique element of $\mathbb{C}^{n\times m}$ which satisfies

$$AA^\dagger A = A,\ A^\dagger AA^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger \text{ and } (A^\dagger A)^* = A^\dagger A.$$

In the case that $m = n$, let $tr(A)$ and I_m denote the trace of $A \in \mathbb{C}^{m\times m}$ and the identity matrix of $\mathbb{C}^{m\times m}$, respectively. An element $P \in \mathbb{C}^{m\times m}$ is said to be an orthogonal projection if $P^2 = P$ and $P^* = P$.

The Moore–Penrose inverse has various applications in controllability problem [4], Markov chain [5], least squares problem [7], linear Glauber model [12], linear Hamiltonian system [13], stochastic signal [14] and so on.

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[1] Supported by the National Natural Science Foundation of China (11171222).
[2] Supported by a foundation of Shanghai Normal University (SK201401).

http://dx.doi.org/10.1016/j.jmaa.2016.01.020

One important research field of the Moore–Penrose inverse is its perturbation theory. In this paper, we deal with norm estimations for the Moore–Penrose inverse associated with rank-preserving perturbations of matrices. Let $A \in \mathbb{C}^{m\times n}$ be given and $B \in \mathbb{C}^{m\times n}$ be a perturbation of A. Clearly, $rank(B) = rank(A)$ if and only if there exist $D_1 \in \mathbb{C}^{m\times m}$ and $D_2 \in \mathbb{C}^{n\times n}$, such that

$$B = D_1^* A D_2, \text{ where } D_1 \text{ and } D_2 \text{ are both nonsingular.} \tag{1.1}$$

In the special case that $\|A^\dagger\|_2 \, \|B - A\|_2 < 1$, it is known that (1.1) holds if and only if $\mathcal{R}(B) \cap \mathcal{R}(A)^\perp = \{0\}$, in which case B is called a stable perturbation of A [17], and norm estimations for $\|B^\dagger - A^\dagger\|_2$ can be found in [8,16,17], respectively.

On the other hand, in the general case the matrix B given by (1.1) is sometimes called a multiplicative perturbation of A, and norm estimations for $\|B^\dagger - A^\dagger\|_F$, as well as $\|B^\dagger - A^\dagger\|_U$ and $\|B^\dagger - A^\dagger\|_Q$ are studied in the literatures [2,9,10,18], where $\|\cdot\|_U$ and $\|\cdot\|_Q$ denote the unitarily invariant norm and the Q-norm [1], respectively. First, an upper bound for $\|B^\dagger - A^\dagger\|_F$ is put forward in [9, Theorem 3.1]. Later, upper bounds for $\|B^\dagger - A^\dagger\|_U$ and $\|B^\dagger - A^\dagger\|_Q$ are figured out in [2, Theorems 4.1 and 4.2], respectively. The results obtained in [2,9] are improved in [18, Theorems 3.1, 3.3 and 3.5], which are improved further in [10, Theorems 2.1, 3.1 and 3.2].

Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ defined by (1.1). In this paper we focus on the study of norm estimations for $\|B^\dagger - A^\dagger\|_F$ and $\|B^\dagger - A^\dagger\|_2$, and have managed to derive new upper bounds without using the Singular Value Decomposition (SVD), which serves however as the main tool in [2,9,10,18]. By replacing the function G defined as (3.21) with a smaller function F given by (3.3), we derive two new upper bounds for $\|B^\dagger - A^\dagger\|_2$ in Theorem 3.3 and Corollary 3.4, respectively. This, together with some parameters being put into the estimations both for the Frobenius norm and the spectral norm, leads to the sharpness of the norm upper bounds (2.9), (3.4) and (3.17). Thus, improvements of [10, Theorems 2.1, 3.1 and 3.2] are made in the cases of the Frobenius norm and the spectral norm.

The rest of this paper is organized as follows. In Section 2, we study norm upper bounds for $\|B^\dagger - A^\dagger\|_F$. In Section 3, we study norm upper bounds for $\|B^\dagger - A^\dagger\|_2$. In Section 4, we provide two numerical examples to illustrate the sharpness of the upper bounds (2.9), (3.4) and (3.17). Finally, several results concerning the norm equality of $P - PQ$ and $Q - QP$ are provided in the Appendix.

2. The Frobenius norm upper bounds

Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ defined by (1.1). In this section, we study the Frobenius norm estimations for $\|B^\dagger - A^\dagger\|_F$, and get a new upper bound established in Theorem 2.2, which generalizes the main technique result of [10, Sec. 2]; see Corollary 2.3 below.

First, we recall some well-known results about the trace and the Frobenius norm of matrices. For any $X \in \mathbb{C}^{m\times n}$ and $Y \in \mathbb{C}^{n\times m}$, it holds that

$$|tr(XY)| \le \|X\|_F \, \|Y\|_F, \tag{2.1}$$

$$\|XY\|_F \le \min\{\|X\|_2 \, \|Y\|_F, \|X\|_F \, \|Y\|_2\}. \tag{2.2}$$

If $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ are two orthogonal projections, then for any $M, N \in \mathbb{C}^{m\times n}$, the following equations hold:

$$\|PM + (I_m - P)N\|_F^2 = \|PM\|_F^2 + \|(I_m - P)N\|_F^2, \tag{2.3}$$

$$\|MQ + N(I_n - Q)\|_F^2 = \|MQ\|_F^2 + \|N(I_n - Q)\|_F^2. \tag{2.4}$$