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Analyticity and compactness of semigroups of composition operators

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A R T I C L E I N F O

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A B S T R A C T

This paper provides a complete characterisation of quasicontractive groups and analytic C_0-semigroups on Hardy and Dirichlet space on the unit disc with a prescribed generator of the form $Af = Gf'$. In the analytic case we also give a complete characterisation of immediately compact semigroups. When the analyticity fails, we obtain sufficient conditions for compactness and membership in the trace class. Finally, we analyse the case where the unit disc is replaced by the right-half plane, where the results are drastically different.

1. Introduction

Semigroups of composition operators acting on the Hardy space $H^2(\mathbb{D})$ or the Dirichlet space $\mathcal{D}$ have been extensively studied (see, for example, [3,4,6,12,20,21]).

These are associated with the notion of semiflow (φ_t) of analytic functions mapping the unit disc $\mathbb{D}$ to itself, and satisfying $\varphi_{s+t} = \varphi_s \circ \varphi_t$; here s and t lie either in $\mathbb{R}_+$ or in a sector of the complex plane. It is assumed that the mapping $(t, z) \mapsto \varphi_t(z)$ is jointly continuous. It follows that there exists an analytic function G on $\mathbb{D}$ such that

$$\frac{\partial \varphi_t}{\partial t} = G \circ \varphi_t.$$

A semiflow induces composition operators C_{φ_t} on $H^2(\mathbb{D})$ or $\mathcal{D}$, where $C_{\varphi_t} f = f \circ \varphi_t$. If it is strongly continuous, then it has a densely-defined generator A given by $Af = Gf'$, with G as above. Fuller details are given later.

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In Section 2 we give a characterisation of analytic semigroups in terms of the properties of G, using the complex Lumer–Phillips theorem [1] (this is appropriate, since the semigroup is quasicontractive, as explained below). In addition, we give a complete description of groups of composition operators in terms of the function G.

The theme of Section 3 is compactness, together with Hilbert–Schmidt and trace-class properties. For example, we give sufficient conditions on G for the semigroup to be immediately compact; these are necessary and sufficient (and equivalent to eventual compactness) when the semigroup is analytic. We give examples to illustrate the various possibilities involving the properties of immediate compactness and eventual compactness. Although most of our results are obtained in terms of the properties of G, we are also able to derive results on compactness from the semiflow model $\varphi_t(z) = h^{-1}(e^{-ct}h(z))$. In particular we are able to provide some answers to a question raised by Siskakis [21, Sec. 8] about how the behaviour of such semigroups depends on the properties of h.

Section 4 is concerned with analytic semigroups and groups of composition operators on the half-plane. Such operators are never compact.

2. Analytic semigroups and groups of composition operators

Definition 2.1. Let $(\beta_n)_{n\geq 0}$ be a sequence of positive real numbers. Then $H^2(\beta)$ is the space of analytic functions

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

in the unit disc $\mathbb{D}$ that have finite norm

$$\|f\|_\beta = \left(\sum_{n=0}^{\infty} |c_n|^2 \beta_n^2 \right)^{1/2}.$$

The case $\beta_n = 1$ gives the usual Hardy space $H^2(\mathbb{D})$.

The case $\beta_0 = 1$ and $\beta_n = \sqrt{n}$ for $n \geq 1$ provides the Dirichlet space $\mathcal{D}$, which is included in $H^2(\mathbb{D})$.

The case $\beta_n = 1/\sqrt{n+1}$ produces the Bergman space, which contains $H^2(\mathbb{D})$.

2.1. General properties of semigroups

A C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X is a mapping $T : \mathbb{R}_+ \to \mathcal{L}(X)$ satisfying

$$\begin{cases} & T(0) = I, \\ \forall t, s \geqslant 0, & T(t+s) = T(t) \circ T(s), \\ \forall x \in X, & \lim_{t\to 0} T(t)x = x. \end{cases}$$

A consequence of this definition is the existence of two scalars $w \geqslant 0$ and $M \geqslant 1$ such that for all $t \in \mathbb{R}_+$, $\|T(t)\| \leqslant Me^{wt}$. In particular, if $M = 1$, the semigroup T is said to be quasicontractive. If in addition $w = 0$, T is a contractive semigroup.

A C_0-semigroup T will be called analytic (or holomorphic) if there exists a sector $\Sigma_\theta = \{re^{i\alpha},\ r \in \mathbb{R}_+,\ |\alpha| < \theta\}$ with $\theta \in (0, \frac{\pi}{2}]$ and an analytic mapping $\widetilde{T} : \Sigma_\theta \to \mathcal{L}(X)$ such that $\widetilde{T}$ is an extension of T and

$$\sup_{\xi \in \Sigma_\theta \cap \mathbb{D}} \|\widetilde{T}(\xi)\| < \infty.$$