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A fundamental solution to the time-periodic Stokes equations

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ABSTRACT

The concept of a fundamental solution to the time-periodic Stokes equations in dimension $n \geq 2$ is introduced. A fundamental solution is then identified and analyzed. Integrability and pointwise estimates are established.

1. Introduction

Classically, fundamental solutions are defined for systems of linear partial differential equations in $\mathbb{R}^n$. Specifically, a fundamental solution to the Stokes system ($n \geq 2$)

$$\begin{cases} -\Delta v + \nabla p = f & \text{in } \mathbb{R}^n, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

with unknowns $v : \mathbb{R}^n \to \mathbb{R}^n$, $p : \mathbb{R}^n \to \mathbb{R}$ and data $f : \mathbb{R}^n \to \mathbb{R}^n$, is a tensor-field

$$\varGamma_{\text{Stokes}} := \begin{pmatrix} \varGamma^{\text{S}}_{11} & \dots & \varGamma^{\text{S}}_{1n} \\ \vdots & \ddots & \vdots \\ \varGamma^{\text{S}}_{n1} & \dots & \varGamma^{\text{S}}_{nn} \\ \gamma^{\text{S}}_1 & \dots & \gamma^{\text{S}}_n \end{pmatrix} \in \mathscr{S}'(\mathbb{R}^n)^{(n+1)\times n}$$

that satisfies[1]

$$\begin{cases} -\Delta \varGamma^{\text{S}}_{ij} + \partial_i \gamma^{\text{S}}_j = \delta_{ij}\delta_{\mathbb{R}^n}, \\ \partial_i \varGamma^{\text{S}}_{ij} = 0, \end{cases} \tag{1.2}$$

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[1] We make use of the Einstein summation convention and implicitly sum over all repeated indices.

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where δ_{ij} and $\delta_{\mathbb{R}^n}$ denote the Kronecker delta and delta distribution, respectively. For arbitrary $f \in \mathscr{S}(\mathbb{R}^n)^n$, a solution (v, p) to (1.1) is then given by the componentwise convolution

$$\begin{pmatrix} v \\ p \end{pmatrix} := \Gamma_{\mathrm{Stokes}} * f, \tag{1.3}$$

which at the outset is well-defined in the sense of distributions. In the specific case of the Stokes fundamental solution Γ_{Stokes} above, L^q-integrability and pointwise decay estimates for (v, p) can be established from (1.3). We refer to the standard literature such as [3] and [9] for these well-known results.

The aim of this paper is to identify a fundamental solution to the *time-periodic* Stokes system

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} = f & \text{in } \mathbb{R}^n \times \mathbb{R}, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}, \\ u(x,t) = u(x, t + \mathcal{T}) \end{cases} \tag{1.4}$$

with unknowns $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $\mathfrak{p} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ corresponding to time-periodic data $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ with the same period, that is, $f(x,t) = f(x, t + \mathcal{T})$. Here $\mathcal{T} \in \mathbb{R}$ denotes the (fixed) time-period. Moreover, $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ denote the spatial and time variable, respectively. The main objective is to establish a framework which enables us to define and identify a fundamental solution $\Gamma_{\mathrm{TPStokes}}$ to (1.4) with the property that a solution $(u, \mathfrak{p})$ is given by a convolution

$$\begin{pmatrix} u \\ \mathfrak{p} \end{pmatrix} := \Gamma_{\mathrm{TPStokes}} * f. \tag{1.5}$$

Having obtained this goal, we shall then examine to which extent regularity such as L^q-integrability and pointwise estimates of the solution can be derived from (1.5).

Since time-periodic data $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $(x,t) \to f(x,t)$ are non-decaying in t, a framework based on classical convolution in $\mathbb{R}^n \times \mathbb{R}$ cannot be applied. Instead, we reformulate (1.4) as a system of partial differential equations on the locally compact abelian group $G := \mathbb{R}^n \times \mathbb{R}/\mathcal{T}\mathbb{Z}$. More specifically, we exploit that $\mathcal{T}$-time-periodic functions can naturally be identified with mappings on the torus group $\mathbb{T} := \mathbb{R}/\mathcal{T}\mathbb{Z}$ in the time variable t. In the setting of the Schwartz–Bruhat space $\mathscr{S}(G)$ and corresponding space of tempered distributions $\mathscr{S}'(G)$, we can then define a fundamental solution $\Gamma_{\mathrm{TPStokes}}$ to (1.4) as a tensor-field

$$\Gamma_{\mathrm{TPStokes}} := \begin{pmatrix} \Gamma^{\mathrm{TPS}}_{11} & \dots & \Gamma^{\mathrm{TPS}}_{1n} \\ \vdots & \ddots & \vdots \\ \Gamma^{\mathrm{TPS}}_{n1} & \dots & \Gamma^{\mathrm{TPS}}_{nn} \\ \gamma^{\mathrm{TPS}}_{1} & \dots & \gamma^{\mathrm{TPS}}_{n} \end{pmatrix} \in \mathscr{S}'(G)^{(n+1)\times n} \tag{1.6}$$

that satisfies

$$\begin{cases} \partial_t \Gamma^{\mathrm{TPS}}_{ij} - \Delta \Gamma^{\mathrm{TPS}}_{ij} + \partial_i \gamma^{\mathrm{TPS}}_{j} = \delta_{ij}\delta_G, \\ \partial_i \Gamma^{\mathrm{TPS}}_{ij} = 0 \end{cases} \tag{1.7}$$

in the sense of $\mathscr{S}'(G)$-distributions. A solution to the time-periodic Stokes system (1.4) is then given by (1.5), provided the convolution is taken over the group G.

The aim in the following is to identify a tensor-field $\Gamma_{\mathrm{TPStokes}} \in \mathscr{S}'(G)^{(n+1)\times n}$ satisfying (1.7). We shall describe $\Gamma_{\mathrm{TPStokes}}$ as a sum of the steady-state Stokes fundamental solution Γ_{Stokes} and a remainder part satisfying remarkably good integrability and pointwise decay estimates. It is well-known that the components of the velocity part $\Gamma^{\mathrm{S}} \in \mathscr{S}'(\mathbb{R}^n)^{n\times n}$ and pressure part $\gamma^{\mathrm{S}} \in \mathscr{S}'(\mathbb{R}^n)^n$ of Γ_{Stokes} are functions