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Asymptotic analysis for time harmonic wave problems with small wave number

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A R T I C L E I N F O

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A B S T R A C T

We study the asymptotic behavior of the solution to some time harmonic wave problems when the wave number is taken as a small asymptotic parameter. Our basic strategy is to introduce suitable Lagrangian multipliers into the governing equations, and transforming them into saddle point problems. These saddle point problems are uniformly invertible with respect to the wave number $k \in [0, k_0]$, with k_0 being an arbitrary but fixed positive number. The asymptotic expansion is then derived by the standard regular perturbation technique.

1. Introduction

PDE problems with small asymptotic parameter are ubiquitous in the science and engineering applications. The study of asymptotic behavior for these problems is crucial at least for two reasons, understanding the possible new physics in the asymptotic regime and designing uniformly stable numerical schemes. Generally, if the limiting problem is well-posed, we call these PDE problems regularly perturbed. Otherwise, we call them singularly perturbed. The regularly perturbed problems are easier, since the solution admits a power series expansion which is valid at least when the asymptotic parameter is sufficiently small. It is the singularly perturbed problems which make the analysis more complicated. Even in the linear case, the asymptotic solutions behavior can be very different, strongly depending on the nature of these PDE problems. A correct solution ansatz with respect to the asymptotic parameter is the key ingredient for this kind of investigations. In this paper, we are interested in the time harmonic wave problems with small wave number.

The first problem we consider is the boundary value problem of the Helmholtz equation

$$-\Delta u - k^2 u = f,\ \forall x \in \Omega, \tag{1}$$

$$\partial_n u - iku = g,\ \forall x \in \Gamma, \tag{2}$$

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where $i = \sqrt{-1}$ denotes the imaginary unit, k is the wave number parameter, $\Omega \subset \mathrm{R}^n$ ($n = 2$ or 3) is a bounded connected Lipschitz domain with connected boundary Γ, n denotes the unit outward normal, $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$ are prescribed complex-valued functions. The weak formulation associated with the boundary value problem (1)–(2) is to find $u \in H^1(\Omega)$, such that for all $v \in H^1(\Omega)$ it holds that

$$(\nabla v, \nabla u) - ik < v, u > -k^2(v, u) = (v, f) + < v, g > . \tag{3}$$

Here and hereafter, we define the volume and boundary duals as

$$(v, u) = \int_\Omega \bar{v} u dx, \quad < v, u >= \int_\Gamma \bar{v} u ds.$$

By the Riesz representation theorem, we can define three bounded linear operators $\{A_j\}_{j=0}^2$ from $H^1(\Omega)$ to $H^1(\Omega)$, and an element $b \in H^1(\Omega)$ as

$$(v, A_0 u)_1 = (\nabla v, \nabla u), \tag{4}$$

$$(v, A_1 u)_1 =< v, u >, \tag{5}$$

$$(v, A_2 u)_1 = (v, u), \tag{6}$$

$$(v, b)_1 = (v, f) + < v, g > . \tag{7}$$

In the above, $(\cdot, \cdot)_1$ stands for the standard H^1-inner product, i.e.,

$$(v, w)_1 = (\nabla v, \nabla w) + (v, w).$$

The variational problem (3) can then be written into an equivalent form of operator equation: find $u \in H^1(\Omega)$ such that

$$(A_0 - ikA_1 - k^2 A_2)u = b. \tag{8}$$

The second problem we consider is the boundary value problem of Navier equation

$$-\nabla \cdot \sigma(u) - k^2 u = f, \ \forall x \in \Omega, \tag{9}$$

$$n \cdot \sigma(u) - iku = g, \ \forall x \in \Gamma, \tag{10}$$

where $\Omega \subset \mathrm{R}^n$ ($n = 2$ or 3) is a bounded connected Lipschitz domain with connected boundary Γ, $\sigma(u)$ stands for the stress tensor of the displacement vector field u. For simplicity, we assume that the stress tensor $\sigma(u)$ relates to the strain tensor $\epsilon(u)$ through

$$\epsilon(u) = (\nabla u + (\nabla u)^\dagger)/2, \quad \sigma(u) = \lambda \mathrm{tr}\epsilon(u) I + 2\mu\epsilon(u).$$

In the above, λ and μ denote the Lame's constants. The weak formulation associated with the boundary value problem (9)–(10) is to find $u \in (H^1(\Omega))^n$, such that for all $v \in (H^1(\Omega))^n$ it holds that

$$\lambda(\mathrm{tr}\epsilon(v), \mathrm{tr}\epsilon(u)) + 2\mu(\epsilon(v), \epsilon(u)) - ik < v, u > -k^2(v, u) = (v, f) + < v, g > . \tag{11}$$

By the Riesz representation theorem, we can define three bounded linear operators $\{A_j\}_{j=0}^2$ from $(H^1(\Omega))^n$ to $(H^1(\Omega))^n$, and an element $b \in (H^1(\Omega))^n$ as