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Partial regularity for subquadratic parabolic systems under controllable growth conditions

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A B S T R A C T

In this paper, we prove a regularity result for weak solutions away from singular set of parabolic systems with subquadratic growth condition of the type

$$\partial_t u(z) - \operatorname{div} a(z, u, Du) = B(z, u, Du) \quad \text{for } z = (x, t),$$

where the structure function a satisfies standard ellipticity and growth condition with polynomial growth rate $\frac{2n}{n+2} < p < 2$. The inhomogeneity B satisfies the controllable growth condition

$$|B(z, u, Du)| \le \lambda\Big(1 + |u|^{p^*-1} + |Du|^{p(1-\frac{1}{p^*})}\Big),$$

where $p^* = \frac{(n+2)p}{n+2-p}$, if $n + 2 \neq p$. Here p^* can be any exponent if $n + 2 = p$. The proof is based on the $\mathcal{A}$-caloric approximation lemma to parabolic systems with subquadratic growth.

1. Introduction and statement of the result

We consider a simplified model of electrorheological fluids, where electrorheological fluids are a special kind of viscous liquid. Some dynamicists have developed a model of complex compressible fluids that is

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based on the mechanical properties of electrorheological fluids. Recently, there are relevant equations of this model in [22] such as the following

$$\operatorname{div}(E+F)=0, \quad \operatorname{curl} E=0, \tag{1.1}$$

$$\rho_0\frac{\partial v}{\partial t}-\operatorname{div} S+\rho_0[\nabla v]v+\nabla\phi=\rho_0 f+[\nabla E]P, \quad \operatorname{div} v=0, \tag{1.2}$$

where E, P, ρ_0, v, S, ϕ and f are the electric field, polarization, density, velocity, extra stress, pressure and the mechanical force, respectively. In fact, the formula of the principal operator of this model is

$$\begin{aligned} S=\alpha_{21}\big((1+|D|^2)^{\frac{P-1}{2}}-1\big)E\otimes E+(\alpha_{31}+\alpha_{33}|E|^2)(1+|D|^2)^{\frac{P-2}{2}}D \\ +\alpha_{51}(1+|D|^2)^{\frac{P-2}{2}}(DE\otimes E+E\otimes DE), \end{aligned} \tag{1.3}$$

where α_{ij} are material constants and the material function p depends on the strength of the electric field $|E|^2$ such that

$$1<p_\infty\le p(|E|^2)\le p_0<\infty. \tag{1.4}$$

This is the non-Newtonian fluid that has made mathematician take a keen interest in some standard simplified model such as the following.

Throughout this paper, on a domain $\Omega_T:=\Omega\times(-T,0)$, where $T>0$ and Ω is a bounded smooth domain in $\mathbb{R}^n$ with dimension $n\ge 2$, we consider weak solutions $u:\Omega_T\to\mathbb{R}^N$ of parabolic systems of the type

$$\partial_t u(z)-\operatorname{div} a(z,u(z),Du(z))=B(z,u(z),Du(z)) \quad \text{for } z=(x,t)\in\Omega_T, \tag{1.5}$$

where

$$a:\Omega_T\times\mathbb{R}^N\times\mathbb{R}^{Nn}\to\mathbb{R}^{Nn}, \quad B:\Omega_T\times\mathbb{R}^N\times\mathbb{R}^{Nn}\to\mathbb{R}^N.$$

Here and in the sequel, we denote the space of linear functions $\mathbb{R}^n\to\mathbb{R}^N$ by $\mathbb{R}^{Nn}$, where $N\ge 1$. The scalar product on $\mathbb{R}^{Nn}$ will be written by $\langle\cdot,\cdot\rangle$ or by a single dot for convenience. The structure function a satisfies some standard ellipticity and growth condition with polynomial growth rate $p\in(\frac{2n}{n+2},2)$. For the inhomogeneity B, we consider the controllable growth condition that will be stated precisely in Section 2.

We are interested in partial regularity results for parabolic systems, that is regularity away from a singular set. In the elliptic case, there are some remarkable results of the partial regularity that have been proven, such as [2,6,15,14], also by Shuhong Chen and Zhong Tan [8,7]. However, the partial regularity for general parabolic systems of the type (1.5) was a longstanding open problem until it was solved by Duzaar and Mingione [17], by Duzaar, Mingione and Steffen [18] and also by [16,3,4]. Their proofs are based on the $\mathcal{A}$-caloric approximation method to the parabolic setting. Moreover, the results of partial regularity in the case of degenerate systems are well known such as [11–13,5]. However, this question under controllable growth conditions in the non-degenerate case remained open so far even for nonsingular parabolic systems. In the subquadratic case, Scheven [23] has proven the partial regularity for the parabolic systems under the following condition

$$|B(z,u,Du)|\le\lambda(1+|Du|^2)^{\frac{p}{4}}.$$

The result on partial regularity under the following controllable growth condition has been treated only in the case of $p=2$ by Shuhong Chen and Zhong Tan [9]