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Circulants and critical points of polynomials

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ABSTRACT

We prove that for any circulant matrix C of size $n \times n$ with monic characteristic polynomial $p(z)$, the spectrum of its $(n-1) \times (n-1)$ submatrix C_{n-1} consisting of the first $n-1$ rows and columns of C consists of all critical points of $p(z)$. Using this fact we provide a simple proof for the Schoenberg conjecture recently proved by R. Pereira and S. Malamud. We also prove full generalization of a higher order Schoenberg-type conjecture proposed by M. de Bruin and A. Sharma and recently proved by W.S. Cheung and T.W. Ng in its original form, i.e. for polynomials whose mass centre of roots equals zero. In this particular case, our inequality is stronger than it was conjectured by de Bruin and Sharma. Some Schmeisser's-like results on majorization of critical point of polynomials are also obtained.

1. Introduction

In [20] I. Schoenberg conjectured the following statement

Schoenberg's conjecture. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of a polynomial p of degree $n \geqslant 2$ such that $\sum_{j=1}^{n} \lambda_j = 0$, and let $w_1, w_2, \ldots, w_{n-1}$ be the roots of the derivative p'. Then*

$$\sum_{k=1}^{n-1} |w_k|^2 \leqslant \frac{n-2}{n} \sum_{j=1}^{n} |\lambda_j|^2,$$

where equality holds if, and only if, all λ_j lie on a straight line.

Later M. de Bruin, K. Ivanov, and A. Sharma [7] and independently Katsoprinakis [14] showed that Schoenberg's conjecture is equivalent to the following inequality

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$$\sum_{k=1}^{n-1}|w_k|^2 \leqslant \frac{1}{n^2}\left|\sum_{j=1}^{n}\lambda_j\right|^2 + \frac{n-2}{n}\sum_{j=1}^{n}|\lambda_j|^2, \tag{1.1}$$

which was conjectured to be true for all complex polynomials with no restrictions on their roots, and equality holds if, and only if, all λ_j lie on a straight line. The inequality (1.1) was proved by R. Pereira [18] and independently by S. Malamud [16,17].

In [8] M. de Bruin and A. Sharma proposed the following higher order Schoenberg-type conjecture.

De Bruin and Sharma's conjecture. *Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the roots of a polynomial p of degree $n \geqslant 2$ such that $\sum_{j=1}^{n}\lambda_j = 0$, and let $w_1, w_2, \dots, w_{n-1}$ be the roots of the derivative p'. Then*

$$\sum_{k=1}^{n-1}|w_k|^4 \leqslant \frac{n-4}{n}\sum_{j=1}^{n}|\lambda_j|^4 + \frac{2}{n^2}\left(\sum_{j=1}^{n}|\lambda_j|^2\right)^2,$$

where equality holds if, and only if, all λ_j lie on a straight line passing through the origin of the complex plane.

This conjecture was proved by W. Cheung and T. Ng in [4] with use their approach of companion matrices (they also give an alternative proof of Schoenberg's conjecture). Later, in [5] they generalized this approach by one-rank perturbation technique. Note that in [5], W. Cheung and T. Ng partially rediscover the so-called "one-rank perturbation method" developed by Yu. Barkovsky in his PhD thesis [1] which, unfortunately, was only partially published. Some ideas of this method can be found among problems in the lecture notes [2]. From the point of view of this method it is easy to see that, indeed, the rank one perturbation approach of W. Cheung and T. Ng generalizes the approach of differentiators used by R. Pereira in [18].

S. Malamud [16,17] proved Schoenberg's conjecture by using another technique (with the same underlying ideas, in fact, basing on the same one-rank perturbation method). To prove the conjecture he established the fact saying that for any polynomial p of degree n, there exists a normal matrix A with $p(z) = \det(zI - A)$ such that the spectrum of its $(n-1) \times (n-1)$-submatrix A_{n-1} constructed with the first $n-1$ rows and the first $n-1$ columns consists of the roots of the derivative of p (Proposition 4.2 in [17]).

In this work, we use Barkovsky's one-rank perturbation method (in its simplified Cheung–Ng's form) to prove that circulant matrix is an example to Malamud's Proposition 4.2, thus giving explicit simple examples of differentiators for a certain class of matrices. Note that W. Cheung and T. Ng also constructed a differentiator explicitly. But they started from a diagonal matrix, so their differentiator has a more complicated form, thus as a consequence, to work with such a differentiator is not easy.

It is known [6] that if $p(z)$ is a complex polynomial, then there exists a circulant matrix

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \dots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \dots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \dots & c_{n-1} & c_0 \end{pmatrix} \tag{1.2}$$

whose characteristic polynomial (up to a constant factor) is $p(z)$. It turned out (Theorem 2.1) that the derivative $p'(z)$ is the characteristic polynomial (up to a constant factor) of the submatrix C_{n-1} constructed with the first $n-1$ rows and columns of C.

Remark 1.1. After this paper was submitted, we learned that the result of Theorem 2.1 was briefly mentioned in Section 6.3 of the work [15] without a detailed proof.