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Is it possible to determine a point lying in a simplex if we know the distances from the vertices?

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ABSTRACT

It is an elementary fact that if we fix an arbitrary set of $d+1$ affine independent points $\{p_0, \ldots, p_d\}$ in $\mathbb{R}^d$, then the Euclidean distances $\{|x - p_j|\}_{j=0}^d$ determine the point x in $\mathbb{R}^d$ uniquely. In this paper we investigate a similar problem in general normed spaces which is motivated by this known fact. Namely, we characterize those, at least d-dimensional, real normed spaces $(X, \|\cdot\|)$ for which every set of $d+1$ affine independent points $\{p_0, \ldots, p_d\} \subset X$, the distances $\{\|x - p_j\|\}_{j=0}^d$ determine the point x lying in the simplex $\mathrm{Conv}(\{p_0, \ldots, p_d\})$ uniquely. If $d = 2$, then this condition is equivalent to strict convexity, but if $d > 2$, then surprisingly this holds only in inner product spaces. The core of our proof is some previously known geometric properties of bisectors. The most important of these (Theorem 1) is re-proven using the fundamental theorem of projective geometry.

1. Introduction

Let X be a real normed space with norm $\|\cdot\|$, and $R, S \subseteq X$. We call R a resolving set for S if for any $s_1, s_2 \in S$, the equations $\|r - s_1\| = \|r - s_2\|$ $(r \in R)$ imply $s_1 = s_2$. We also say that R resolves S, and in the literature this notion is also referred to as metric generator or determining set. This quite natural notion originally was defined for general metric spaces in [7] in 1953, but it attracted little attention at that time. In the theory of finite dimensional normed spaces (or Minkowski spaces) this notion was investigated in [22]. Namely, Kalisch and Straus proved that a d-dimensional normed space is Euclidean if and only if every subset A which is not contained in a hyperplane is a resolving set for the whole space. We note that as a consequence of our results, we will strengthen this theorem in Corollary 3.

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In 1975, a very similar concept was introduced in graph theory [16,30]. Since then, several papers have been published concerning this direction (see e.g. [2,3,8,9,15,17,20] for some recent developments), and this topic has a wide range of applications for instance in informatics, robotics, biology and chemistry (see e.g. [10,13,26]).

The notion of resolving sets in metric spaces is naturally related to the characterization of isometries of certain metric spaces. Recently, motivated by some problems in quantum mechanics, this notion was implicitly used in [27–29] in order to describe isometries of certain classes of matrices. Furthermore, with the help of resolving sets, the author of the present paper gave a new and elementary proof of a famous theorem of Wigner, which is very important in the foundation of quantum physics (see [14]).

Also recently, motivated by complex analysis, some basic results on resolving sets in general metric spaces were provided in [4]. The metric dimensions of the hypercube in $\mathbb{R}^d$ and some important geometric spaces were discussed in [5,18].

In this paper we will consider general real normed spaces. For a number $d \in \mathbb{N}$, $d \geq 2$, we say that a normed space X with $\dim X \geq d$ has the property (SRSd), if it satisfies the following condition:

$$\textit{every set of } d+1 \textit{ affine independent points resolves its convex hull.} \tag{SRSd}$$

(Here SRS stands for "simplex resolving set".) In this paper we are interested in the problem of characterizing those normed spaces which satisfy the property (SRSd). It will turn out that those at least two-dimensional normed spaces X in which (SRS2) holds are precisely the strictly convex spaces (Theorem 2). After that one would expect that we obtain the same conclusion if we consider (SRSd) with $d \geq 3$, since there is no immediate reason which suggests otherwise. But on the contrary, when $d \geq 3$, an at least d-dimensional space X satisfies (SRSd) if and only if it is an inner product space (Theorem 3).

The characterization of strictly convex and inner product spaces is a classical field of functional analysis. There are several characterizations, and several of them were collected in the book of Amir [1] (see also e.g. [11,25,31,32] concerning some recent results). We emphasize the well-known Jordan–von Neumann theorem which was proven originally for complex spaces in [21], but as was pointed out there, real spaces can be handled along the same lines (even with some simplifications). It states that a normed space X is an inner product space if and only if its norm $\|\cdot\|$ satisfies the parallelogram identity:

$$2(\|x\|^2 + \|y\|^2) = \|x+y\|^2 + \|x-y\|^2 \quad (x, y \in X). \tag{1}$$

This theorem further implies the following, which was also mentioned in [21]: if $\dim X > 2$ ($\dim X > 3$, respectively) and the restriction of the norm to any (linear) subspace with dimension two (three, resp.) is Euclidean, then the norm of X comes from an inner product as well. Obviously, a similar conclusion holds for strict convexity, which is straightforward from its definition. In other words, strict convexity and inner productness are two-dimensional properties. We also note that if one replaces the equality sign in (1) by "$\geq$", then this inequality still characterizes inner productness (see [25]).

From now on, if we do not say otherwise, X will always denote a real normed space with norm $\|\cdot\|$. Whenever we say subspace, we will mean a linear subspace. On the other hand, when we consider a translated copy of a subspace, we will call it an affine subspace. In particular, we will say line if it is one-dimensional, plane if it is two-dimensional, or hyperplane if its codimension is one. We shall make use of the following notation: for every two points $x, y \in X$, $x \neq y$ let

$$B(x,y) := \{z \in X : \|z-x\| = \|z-y\|\} \subseteq X,$$

which is usually called the bisector of x and y. Geometric properties of bisectors in finite dimensional normed spaces yield various deep characterizations of special normed spaces (see e.g. the survey papers [24, 23]). This notion is also naturally related to the study of Voronoi diagrams. We clearly have $B(x,y) =$