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Dirichlet problem for anisotropic prescribed mean curvature equation on unbounded domains

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ABSTRACT

In this paper, we consider the Dirichlet problem for hypersurfaces $\mathcal{M} = graph\, u$ of anisotropic prescribed mean curvature $H = H(x, u, N)$ on unbounded domain Ω, where N is the unit normal to $\mathcal{M}$ at (x, u). As a corollary of the result, we obtain the existence of translating solutions to the mean curvature flow with a forcing term on unbounded domains. The approach used here is a modified version of classical Perron's method, where the solutions to minimal surface equation are used as supersolutions and a family of auxiliary functions is constructed as local subsolutions.

1. Introduction and main results

Let Ω be a domain in $\mathbb{R}^n$ $(n \geq 2)$, a function $\varphi \in C^0(\partial\Omega)$ and let $\mathcal{M}$ be a hypersurface over Ω given as graph of $u : \Omega \to \mathbb{R}$. We consider the Dirichlet problem:

$$\begin{cases} \frac{1}{n} div\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = H(x, u, N(Du)), & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $N(Du) = \frac{1}{\sqrt{1+|Du|^2}}(-Du, 1)$ is the upward unit normal vector field of the hypersurface $\mathcal{M}$.

When Ω is a bounded domain, Serrin [17] solved first the Dirichlet problem (1.1) with the prescribed mean curvature $H = \Lambda(x) \in C^1(\bar{\Omega})$. The result in [17] is that there exists a solution $u \in C^{2,\alpha}(\bar{\Omega})$ provided $\partial\Omega \in C^{2,\alpha}$, $\varphi \in C^{2,\alpha}(\bar{\Omega})$ and the mean curvature $H_{\partial\Omega}$ on the boundary $\partial\Omega$ satisfying

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$$H_{\partial\Omega}(x) \geq \frac{n}{n-1}|\Lambda(x)|, \quad \forall\, x \in \partial\Omega, \tag{1.2}$$

as well as Marquardt considered in [15] the general case $H = H(x, u, N(Du))$, see the following theorem.

Theorem 1.1. *(See [15].) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$ boundary, $\varphi \in C^0(\partial\Omega)$. If $H(x, z, N)$ satisfies*

$$\int_\Omega \sup_{z\in\mathbb{R}, N\in\mathbb{S}^n} |H(x, z, N)|^n dx < \omega_n, \tag{1.3}$$

and $H(x, z, N)$ can be written as $H(x, z, N) = H_1(x, z, N) + H_2(x, z, N)N_{n+1}$ on $\Omega\times\mathbb{R}\times\mathbb{S}^n$, where $H_1, H_2 \in C^1(\bar\Omega \times \mathbb{R} \times \mathbb{S}^n) \cap C^{1,\gamma}(\Omega \times \mathbb{R} \times \mathbb{S}^n)$ $(0 < \gamma < 1)$, $D_z H_1 \geq 0$ and

$$H_{\partial\Omega}(x) \geq \frac{n}{n-1}|H_1(x, \varphi(x), \gamma_0(x))|, \quad \forall\, x \in \partial\Omega, \tag{1.4}$$

then the Dirichlet problem (1.1) has a solution $u \in C^0(\bar\Omega) \cap C^2(\Omega)$. In the case $D_z H \geq 0$, (1.3) can be replaced by the following condition

$$\left|\int_\Omega H(x, 0, N(D\eta))dx\right| < \frac{1-\varepsilon}{n}\int_\Omega |D\eta|dx, \quad \forall\, \eta \in C_0^1(\Omega) \tag{1.5}$$

for some $\varepsilon > 0$ ensure a bound on u. Furthermore, the solution is unique if $D_z H \geq 0$.

Here and below, $H_{\partial\Omega}$ denotes the mean curvature of $\partial\Omega$ with respect to the inner normal, γ_0 is the inward pointing unit normal of $\partial\Omega$, and N_{n+1} is the last coordinate component of normal vector N.

If $H(x, z, N) = \Lambda(x)$ for some C^1 function $\Lambda(x)$, Jin investigated in [9] the existence of solutions to (1.1) on unbounded domains under the assumptions that $0 \leq \Lambda(x) < \frac{n-1}{nM}$ on $\bar\Omega$ and $H_{\partial\Omega}(y) > \frac{n}{n-1}\Lambda(y)$ on $\partial\Omega$, where M is a constant depending on domain Ω. The method used in [9] is a modified Perron's method, where the solutions to the minimal surface equation are used as subsolutions and a family of auxiliary functions is constructed as local supersolutions. There are similar Dirichlet problems on unbounded domains for constant mean curvature surfaces in [14] and the equation of translating solution to the powers of mean curvature flow in [10].

Motivated by [15,9,14,10], we consider the Dirichlet problem (1.1) with more general anisotropic prescribed mean curvature $H = H(x, u, N(Du))$ on more general domains. We mainly refer to the method used in [9,10]. Before stating the main results, we give some assumptions for Ω and $H(x, z, N)$ as follows.

Assumptions for Ω:

(Ω_1) $\partial\Omega \in C^{2,\alpha}$ with $0 < \alpha < 1$;

(Ω_2) There are constants N and $M > 0$ such that

$$\Omega \subset C_N(M) := \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_1 > N, x_2^2 + \cdots x_n^2 < M^2\}.$$

Assumptions for $H(x, z, N)$:

(H_0) $H(x, z, N)$ can be written as $H(x, z, N) = H_1(x, z, N) + H_2(x, z, N)N_{n+1}$ on $\Omega \times \mathbb{R} \times \mathbb{S}^n$, where $H_1, H_2 \in C^1(\bar\Omega \times \mathbb{R} \times \mathbb{S}^n) \cap C^{1,\gamma}(\Omega \times \mathbb{R} \times \mathbb{S}^n)$ $(0 < \gamma < 1)$, $D_z H_1 \geq 0$ and H_2 is a bounded function, i.e., there is a constant $M_0 > 0$ such that $|H_2(x, z, N)| \leq M_0$ for $(x, z, N) \in \bar\Omega \times \mathbb{R} \times \mathbb{S}^n$.

Our main result is in the following: