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Existence and uniqueness of limit cycles for generalized φ-Laplacian Liénard equations

S. Pérez-González [a,*], J. Torregrosa [b], P.J. Torres [c]

[a] Departamento de Matemática, Universidade Estadual Paulista Júlio de Mesquita Filho, Rua Cristovão Colombo, 2265, 15054-000 São José do Rio Preto, SP, Brazil
[b] Departament de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C 08193 Bellaterra, Barcelona, Spain
[c] Departamento de Matemática Aplicada, Universidad de Granada, 18071 Granada, Spain

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ABSTRACT

The Liénard equation $x'' + f(x)x' + g(x) = 0$ appears as a model in many problems of science and engineering. Since the first half of the 20th century, many papers have appeared providing existence and uniqueness conditions for limit cycles of Liénard equations. In this paper we extend some of these results for the case of the generalized φ-Laplacian Liénard equation, $(\varphi(x'))' + f(x)\psi(x') + g(x) = 0$. This generalization appears when derivations of the equation different from the classical one are considered. In particular, the relativistic van der Pol equation, $\left(x'/\sqrt{1-(x'/c)^2}\right)' + \mu(x^2-1)x' + x = 0$, has a unique periodic orbit when $\mu \neq 0$.

1. Introduction

The Liénard equation,

$$x'' + f(x)x' + g(x) = 0, \tag{1}$$

appears as a model in many areas in science and engineering. It was intensively studied during the first half of the 20th century as it can be used to model oscillating circuits or simple pendulums. In the case of the simple pendulum, the functions f and g represent the friction and acceleration terms. One of the first models where this equation appeared was introduced by van der Pol [12], considering the equation

$$x'' + \mu(x^2 - 1)x' + x = 0,$$

for modeling the oscillations of a triode vacuum tube. See [7] for other references about more applications.

* Corresponding author.
E-mail addresses: setperez@mat.uab.cat (S. Pérez-González), torre@mat.uab.cat (J. Torregrosa), ptorres@ugr.es (P.J. Torres).

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The first results on existence and uniqueness of periodic solutions of Liénard equations appear in [8,13]. More references on the existence and uniqueness of limit cycles are [16–19]. Additionally, [3] and [9] deal with related problems.

In this work, new criteria are presented for existence and uniqueness results of isolated periodic orbits (limit cycles) for the generalized φ-Laplacian Liénard equation

$$(\varphi(x'))' + f(x)\psi(x') + g(x) = 0. \tag{2}$$

Besides the obvious mathematical interest of this generalization, our main motivation for considering this equation comes from relativistic models. Special Relativity imposes a universal bound for the propagation speed of any gravitational or electromagnetic wave. If c is the speed of light in vacuum, in the framework of Special Relativity the momentum of a particle with unitary rest mass is given by $\varphi(x') = x'/\sqrt{1-(x'/c)^2}$, see for instance [6]. The results of the present paper ensure the existence and uniqueness of a periodic solution for the relativistic van der Pol equation

$$\left(\frac{x'}{\sqrt{1-(x'/c)^2}}\right)' + \mu(x^2-1)x' + x = 0,$$

when $\mu \neq 0$. The harmonic relativistic oscillator case, $\mu = 0$, is a classical topic studied by several authors, see for instance [5,11]. Other authors have considered the existence of periodic solutions of damped oscillators with non-autonomous non-linear forces, see [14,15].

The Liénard equation (1) is commonly expressed as the planar system

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -g(x), \end{cases} \tag{3}$$

where $F(x) = \int_0^x f(s)ds$, or

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -g(x) - f(x)y. \end{cases}$$

Writing equation (2) as the system

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \varphi'(y)\dfrac{dy}{dt} = -g(x) - f(x)\psi(y), \end{cases}$$

we can consider the equivalent system

$$\begin{cases} \dot{x} = y\varphi'(y), \\ \dot{y} = -g(x) - f(x)\psi(y), \end{cases} \tag{4}$$

after the time rescaling $d\tau = dt/\varphi'(y)$, whenever $\varphi(y)$ is of class $\mathcal{C}^{1,1}$ and $\varphi'(y) \neq 0$ for all y, where $\dot{x}$, $\dot{y}$ denote the derivatives of x, y with respect to τ. In general, system (4) cannot be transformed to system (3). Hence the classical results do not apply to the equation proposed in this paper.

Before stating the results, some necessary hypotheses are introduced. All the functions in (2) should be at least locally Lipschitz continuous, $\mathcal{C}^{0,1}$, except for $\varphi(y)$ which should be in $\mathcal{C}^{1,1}$. These properties ensure the existence and uniqueness of a solution for any initial value problem associated with system (4). Further regularity conditions of each function are required for more specific results.