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The demiclosedness principle for mean nonexpansive mappings

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ABSTRACT

In this paper, we establish the demiclosedness principle for the class of mean nonexpansive mappings, introduced in 2007 by Goebel and Japón Pineda, defined on closed, convex subsets of Banach spaces satisfying Opial's condition. We also establish the demiclosedness principle for a subclass of the mean nonexpansive maps whose domains are closed, bounded, convex sets in uniformly convex spaces. These results extend known demiclosedness results for nonexpansive maps and lead to some fixed point results for mean nonexpansive maps which partially answer an open question posed by Goebel and Japón Pineda.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space, and C a nonempty subset of X. A function $T : C \to X$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|\,, \text{ for all } x, y \in C.$$

It is a well-known application of Banach's Contraction Mapping Principle that every nonexpansive mapping $T : C \to C$ (where C is closed, bounded, and convex) has an *approximate fixed point sequence* $(x_n)_n$ in C. That is, $(x_n)_n$ is a sequence for which $\|Tx_n - x_n\| \to 0$. The question of when nonexpansive maps have fixed points is much more delicate, however. We say a Banach space $(X, \|\cdot\|)$ has *the fixed point property for nonexpansive maps* if, for every closed, bounded, convex subset $C \neq \emptyset$ of X, every nonexpansive map $T : C \to C$ has a fixed point (that is, a point $x \in C$ for which $Tx = x$).

Recall that a Banach space $(X, \|\cdot\|)$ is called *uniformly convex* if, for all $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ for which

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$$\begin{cases} \|x\| \leq 1, \\ \|y\| \leq 1, \\ \|x-y\| \geq \varepsilon \end{cases} \implies \frac{1}{2}\|x+y\| \leq 1-\delta.$$

In 1965, Browder [1], Göhde [8], and Kirk [9] independently proved that uniformly convex Banach spaces have the fixed point property for nonexpansive maps (and, in the case of Kirk, that the more general class of reflexive Banach spaces with normal structure has the fixed point property). As the central ingredient of his proof, Göhde showed that, whenever $(X, \|\cdot\|)$ is uniformly convex, $C \subset X$ is closed, bounded, and convex, and $T : C \to X$ is nonexpansive, it follows that $I - T$, where I denotes the identity map on X, is demiclosed. In 1967, Opial [14] also proved the demiclosedness of nonexpansive maps defined on closed, convex sets in spaces satisfying Opial's property, while in 1968, Browder [2] extended Göhde's result to the class of semicontractive maps on uniformly convex spaces (we will discuss these notions in detail in the next section).

In this paper, we will extend the demiclosedness principle to the class of so-called "mean nonexpansive maps," which were introduced in 2007 by Goebel and Japón Pineda [5]. More specifically, in Theorems 3.1 and 3.4, we establish the demiclosedness principle for (α, p)-nonexpansive maps $(1 < p < \infty)$ in uniformly convex spaces. In Theorem 4.2, we establish the demiclosedness principle for α-nonexpansive maps defined on closed, convex sets satisfying Opial's property. We then use these theorems to obtain fixed point results (Theorem 5.1) for mean nonexpansive maps, which rely on the existence of an approximate fixed point sequence and partially answer open questions posed by Goebel and Japón Pineda.

2. Preliminaries

A function $T : C \to C$ is called *mean nonexpansive* (or *α-nonexpansive*) if, for some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\sum_{k=1}^{n} \alpha_k = 1$, $\alpha_k \geq 0$ for all k, and $\alpha_1, \alpha_n > 0$, we have

$$\sum_{k=1}^{n} \alpha_k \left\|T^k x - T^k y\right\| \leq \|x - y\|\,, \text{ for all } x, y \in C.$$

Goebel and Japón Pineda further suggested the class of (α, p)-nonexpansive maps. A function $T : C \to C$ is called *(α, p)-nonexpansive* if, for some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\sum_{k=1}^{n} \alpha_k = 1$, $\alpha_k \geq 0$ for all k, $\alpha_1, \alpha_n > 0$, and for some $p \in [1, \infty)$,

$$\sum_{k=1}^{n} \alpha_k \left\|T^k x - T^k y\right\|^p \leq \|x - y\|^p\,, \text{ for all } x, y \in C.$$

For simplicity, we will generally discuss the case when $n = 2$. That is, $T : C \to C$ is *$((\alpha_1, \alpha_2), p)$-nonexpansive* if for some $p \in [1, \infty)$, we have

$$\alpha_1 \|Tx - Ty\|^p + \alpha_2 \left\|T^2 x - T^2 y\right\|^p \leq \|x - y\|^p\,, \text{ for all } x, y \in C.$$

When $p = 1$, we will say T is (α_1, α_2)-nonexpansive. When the multi-index α is not specified, we say T is *mean nonexpansive.*

As one can imagine, (α, p)-nonexpansive maps are natural to study in L^p spaces. The following is an example of a $(\frac{1}{2}, \frac{1}{2})$-2-nonexpansive map defined on $(\ell^2, \|\cdot\|_2)$ for which none of its iterates are nonexpansive. The map below is based on an example given by Goebel and Sims [11].

Example 2.1. Let $(\ell^2, \|\cdot\|_2)$ be the Hilbert space of square-summable sequences endowed with its usual norm. Let $\tau : [-1, 1] \to [-1, 1]$ be given by