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Asymptotic behavior for the critical nonhomogeneous porous medium equation in low dimensions

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We deal with the large time behavior for a porous medium equation posed in nonhomogeneous media with singular critical density

$$|x|^{-2}\partial_t u(x,t) = \Delta u^m(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,\infty),\ m \geq 1,$$

posed in dimensions $N = 1$ and $N = 2$, which are also interesting in applied models according to works by Kamin and Rosenau. We deal with the Cauchy problem with bounded and continuous initial data u_0. We show that in dimension $N = 2$, the asymptotic profiles are self-similar solutions that vary depending on whether $u_0(0) = 0$ or $u_0(0) = K \in (0,\infty)$. In dimension $N = 1$, things are strikingly different, and we find new asymptotic profiles of an unusual mixture between self-similar and traveling wave forms. We thus complete the study performed in previous recent works for the higher dimensions $N \geq 3$.

1. Introduction

The goal of this work is to study the asymptotic behavior of solutions to the following nonhomogeneous porous medium equation with critical singular density

$$|x|^{-2}\partial_t u(x,t) = \Delta u^m(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \tag{1.1}$$

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where $m \geq 1$ and we restrict ourselves to dimensions $N = 1$ and $N = 2$. This completes the panorama of the large time behavior of solutions to Eq. (1.1), already studied in dimensions $N \geq 3$ in previous recent works by the authors [18,19]. We will work with the Cauchy problem for initial data satisfying

$$u(x, 0) = u_0(x) \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N), \; u_0 \geq 0, \; u_0 \not\equiv 0, \quad \text{for any } x \in \mathbb{R}^N. \tag{1.2}$$

Moreover, unless we explicitly state the contrary, we will always consider *radially symmetric* initial conditions, that is, $u_0(x) = u_0(r)$, $r = |x|$. The radial symmetry of solutions is an essential restriction in dimension $N = 2$, since the well-posedness and regularity theory is only established for radially symmetric data, see [25]. It is not our aim to try to extend this theory to non-symmetric data here, but rather to focus on the dynamics of the equation. However, in dimension $N = 1$, this restriction is readily removed, see Part 2 in Section 5.

Equations such as

$$\varrho(x)\partial_t u(x, t) = \Delta u^m(x, t), \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \tag{1.3}$$

where ϱ is a density function with suitable behavior, have been proposed by Kamin and Rosenau in several classical papers [21–23] as a model for thermal propagation by radiation in non-homogeneous plasma. Afterwards, a big development of the mathematical theory associated to Eq. (1.3) begun, assuming that the density satisfies

$$\varrho(x) \sim |x|^{-\gamma}, \quad \text{as } |x| \to \infty,$$

for some $\gamma > 0$, as for example in the following papers [9,10,12,25,32,36,33,34,20,28] where its qualitative properties and asymptotic behavior are studied. In particular, some of these works were also considering purely singular densities such as $\varrho(x) = |x|^{-\gamma}$, and proving that asymptotic profiles for (1.3) come from explicit solutions to equations with singular density. Moreover, it came out that the density $\varrho(x) = |x|^{-2}$ (or $\varrho(x) \sim |x|^{-2}$ as $|x| \to \infty$) is critical for both the qualitative properties (well-posedness, regularity) and large time behavior: indeed, for $\gamma \in (0, 2)$, properties do not depart from those already well investigated of the porous medium equation $u_t = \Delta u^m$, while for $\gamma > 2$, they are different.

Going back to the singular case $\varrho(x) = |x|^{-\gamma}$, its theory developed later, due to the difficulties involved by the presence of the singular coefficient. Some results on qualitative behavior were established in [25], then later in [20, Section 6] and [28, Section 3], the latter two using weighted spaces for well-posedness, and restricting the study to dimensions $N \geq 3$. On the other hand, formal transformations and explicit solutions were established in [12,17], where it becomes clear why $\gamma = 2$ is a critical exponent. Restricting to (1.1), the large time behavior in dimensions $N \geq 3$ has been established recently by the authors [18,19], and the results were quite striking: the presence of both critical density $\gamma = 2$ and the singularity at $x = 0$ led to many new and unexpected mathematical phenomena, that in the general, non-critical and/or non-singular case do not happen. We recently learned that also the nonhomogeneous equation for the fractional porous medium

$$\varrho(x)\partial_t u = (-\Delta)^s(u^m), \quad m > 1, \; s \in (0, 1),$$

has been proposed in [15,16], where the qualitative properties and large time behavior are studied for suitable $\varrho(x)$, but again excluding the critical density (that in the fractional case holds for $\gamma = 2s$). More general, weighted forms of the porous medium equation (involving even two different weights) have been proposed in [31,14]. Finally, we found out that the theory for the nonhomogeneous porous medium equation with critical density $\varrho(r) \sim r^{-2}$ as $r \to \infty$ found an interesting application in the study of a porous medium equation in the hyperbolic space, see [40].