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Sharp eigenvalue estimates for rank one perturbations of nonnegative operators in Krein spaces

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A B S T R A C T

Let A and B be selfadjoint operators in a Krein space and assume that the resolvent difference of A and B is of rank one. In the case that A is nonnegative and I is an open interval such that $\sigma(A) \cap I$ consists of isolated eigenvalues we prove sharp estimates on the number and multiplicities of eigenvalues of B in I. The general result is illustrated with eigenvalue estimates for singular indefinite Sturm–Liouville problems.

1. Introduction

Rank one and finite rank perturbations of selfadjoint operators in Hilbert spaces have been considered in various papers and in many applications in theoretical physics, e.g. in the investigation of singular perturbations in quantum mechanics, see [1–3,11,23,24,28,31,32,45,55]. It is well known that an n-dimensional selfadjoint perturbation of a selfadjoint operator in a Hilbert space preserves the essential spectrum and changes the spectral multiplicity by at most n, that is, for a bounded interval $I \subset \mathbb{R}$ and (in general unbounded) selfadjoint operators A, B in a Hilbert space $\mathscr{H}$ such that

$$(A - \lambda_0)^{-1} - (B - \lambda_0)^{-1} \tag{1.1}$$

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is of rank n for some $\lambda_0 \in \rho(A) \cap \rho(B)$, the dimensions of the spectral subspaces of A and B corresponding to the interval I differ at most by n, and this estimate is sharp. In particular, if $I \subset \rho(A)$ then I contains at most n eigenvalues of B counted with multiplicities.

In the general non-selfadjoint case rank one and finite rank perturbations preserve the essential spectrum but precise results on the number and multiplicity of the discrete spectrum do not exist. Without further assumptions on the structure of the operators or the rank one perturbation the number of eigenvalues in a given interval can change arbitrarily, see [44, Theorem 1]. If the operators A and B under consideration are not selfadjoint in a Hilbert space but still selfadjoint in a Krein space, then several results on finite rank perturbations of different classes of operators exist; cf. [4,5,7,8,13,22,26,34–37]. However, these perturbation results are typically of qualitative nature and do not contain explicit bounds or estimates on the number and multiplicities of eigenvalues after the perturbation. In the matrix case we refer to [51–53], where so-called generic perturbations were investigated, and in [54] some estimates and bounds in the case of a Pontryagin space are given.

Our main objective in this paper is to obtain sharp bounds for the number and multiplicities of eigenvalues in the following Krein space perturbation problem: We assume that A and B are selfadjoint with respect to some indefinite inner product $[\cdot,\cdot]$, that A is nonnegative with respect to $[\cdot,\cdot]$, and that the perturbation (1.1) is of rank one. In that case B is either nonnegative (and we write $\kappa_B = 0$) or the form $[B\cdot,\cdot]$ has one negative square (and we write $\kappa_B = 1$). Let I be an open interval such that all spectral points of A in I are isolated eigenvalues and poles of the resolvent of A; here also eigenvalues of infinite multiplicity are allowed. In this setting our first main result (Theorem 3.5 below) states: The difference of the number $n_A(I)$ of distinct eigenvalues of A in I and the number $n_B(I)$ of distinct eigenvalues of B in I can be estimated by the number $n_{A,B}(I)$ of common eigenvalues of A and B in I, and a correction term which is at most 3. The correction term depends on the fact whether 0 is in the interval I and whether the operator B is nonnegative ($\kappa_B = 0$) or has one negative square ($\kappa_B = 1$):

(i) If $0 \notin I$ then

$$n_A(I) - n_{A,B}(I) - 1 \leq n_B(I) \leq n_A(I) + n_{A,B}(I) + \begin{cases} 1 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

(ii) If $0 \in I$ then

$$n_A(I) - n_{A,B}(I) - 2 \leq n_B(I) \leq n_A(I) + n_{A,B}(I) + \begin{cases} 2 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

It is remarkable that all the above estimates turn out to be sharp: There exist operators A and B (which are in fact matrices) such that the inequalities in (i) and (ii) become equalities. Moreover, we mention that the above estimates imply that the finiteness of the number of distinct eigenvalues of A in a gap of the essential spectrum is preserved under a one dimensional perturbation. This is a special case of a more general result from [13].

Our second main result are estimates of the total algebraic multiplicities $m_A(I)$ and $m_B(I)$ of the eigenvalues of A and B in I. This leads to the following estimates in Theorem 3.9 on the multiplicities of the eigenvalues which complement the results in Theorem 3.5 on the number of distinct eigenvalues:

(i) If $0 \notin I$ then

$$m_A(I) - 1 \leq m_B(I) \leq m_A(I) + \begin{cases} 1 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$