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Asymptotic estimate of eigenvalues of pseudo-differential operators in an interval ☆

Kamil Kaleta, Mateusz Kwaśnicki *, Jacek Małecki

Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, ul. Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

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ABSTRACT

We prove a two-term Weyl-type asymptotic law, with error term $O(\frac{1}{n})$, for the eigenvalues of the operator $\psi(-\Delta)$ in an interval, with zero exterior condition, for complete Bernstein functions ψ such that $\xi\psi'(\xi)$ converges to infinity as $\xi \to \infty$. This extends previous results obtained by the authors for the fractional Laplace operator ($\psi(\xi) = \xi^{\alpha/2}$) and for the Klein–Gordon square root operator ($\psi(\xi) = (1+\xi^2)^{1/2} - 1$). The formula for the eigenvalues in $(-a, a)$ is of the form $\lambda_n = \psi(\mu_n^2) + O(\frac{1}{n})$, where μ_n is the solution of $\mu_n = \frac{n\pi}{2a} - \frac{1}{a}\vartheta(\mu_n)$, and $\vartheta(\mu) \in [0, \frac{\pi}{2})$ is given as an integral involving ψ.

1. Introduction and statement of the results

This is the final one in the series of articles where asymptotic formulae for eigenvalues of certain pseudo-differential operators in the interval are studied. The fractional Laplace operator $(-\Delta)^{\alpha/2}$ was considered in [20] for $\alpha = 1$ and in [22] for general $\alpha \in (0, 2)$, while in [18] the case of the Klein–Gordon square-root operator $(-\Delta + 1)^{1/2} - 1$ was solved (Δ dentotes the second derivative operator, the Laplace operator in dimension one). In the present article we extend the above results to operators $\psi(-\Delta)$, where ψ is an arbitrary complete Bernstein function such that $\xi\psi'(\xi)$ converges to infinity as $\xi \to \infty$.

Let λ_n denote the nondecreasing sequence of eigenvalues of $\psi(-\Delta)$ in an interval $D = (-a, a)$, with zero condition in the complement of D. Furthermore, for $\mu > 0$ define

$$\vartheta_\mu = \frac{1}{\pi} \int_0^\infty \frac{\mu}{r^2 - \mu^2} \log \frac{\psi'(\mu^2)(\mu^2 - r^2)}{\psi(\mu^2) - \psi(r^2)} \, dr. \tag{1}$$

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* Corresponding author.
E-mail addresses: kamil.kaleta@pwr.edu.pl (K. Kaleta), mateusz.kwasnicki@pwr.edu.pl (M. Kwaśnicki), jacek.malecki@pwr.edu.pl (J. Małecki).

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We note that $\vartheta_\mu \in [0, \frac{\pi}{2})$ and $\frac{d}{d\mu}\vartheta_\mu = O(\frac{1}{\mu})$ as $\mu \to \infty$. Finally, let μ_n be a solution of

$$\mu_n = \tfrac{n\pi}{2a} - \tfrac{1}{a}\vartheta_{\mu_n}. \tag{2}$$

We remark that the solution is unique for n large enough, and

$$\mu_n = \tfrac{n\pi}{2a} - \tfrac{1}{a}\vartheta_{(n\pi)/(2a)} + O(\tfrac{1}{n}).$$

The following is the main result of the present article.

Theorem 1.1. *If ψ is a complete Bernstein function and $\lim_{\xi\to\infty} \xi\psi'(\xi) = \infty$, then*

$$\lambda_n = \psi(\mu_n^2) + O(\tfrac{1}{n}) \qquad \text{as } n \to \infty. \tag{3}$$

In many cases, μ_n can be approximated with more explicit expressions, at the price of a weaker estimate of the error term. We provide two examples.

Example 1.2. Let $\psi(\xi) = \xi^{\alpha/2} + \xi^{\beta/2}$, where $0 < \beta < \alpha \le 2$. Then (see Example 2.11)

$$\vartheta_\mu = \tfrac{(2-\alpha)\pi}{8} + O(n^{\beta-\alpha}), \qquad \mu_n = \tfrac{n\pi}{2a} - \tfrac{(2-\alpha)\pi}{8a} + O(n^{\beta-\alpha}),$$

and consequently

$$\lambda_n = (\tfrac{n\pi}{2a} - \tfrac{(2-\alpha)\pi}{8a})^\alpha + (\tfrac{n\pi}{2a} - \tfrac{(2-\alpha)\pi}{8a})^\beta + O(n^{\beta-1}).$$

Example 1.3. If ψ is regularly varying at infinity with index $\frac{\alpha}{2} \in (0, 1]$, then one has $\lim_{\mu\to\infty} \vartheta_\mu = \frac{(2-\alpha)\pi}{8}$ (see (15)). Therefore,

$$\mu_n = \tfrac{n\pi}{2a} - \tfrac{(2-\alpha)\pi}{8a} + o(1),$$

and, using Karamata's monotone density theorem, one easily finds that

$$\lambda_n = (1 - \tfrac{(2-\alpha)\alpha}{4n} + o(\tfrac{1}{n}))\psi((\tfrac{n\pi}{2a})^2).$$

Remark 1.4. The *moderate growth condition* $\lim_{\xi\to\infty} \xi\psi'(\xi) = \infty$ is satisfied by all regularly varying functions with positive index. It is *not* satisfied by a slowly varying complete Bernstein function $\psi(\xi) = \log(1+\xi)$. There are, however, slowly varying functions which do satisfy the moderate growth condition, for example,

$$\psi(\xi) = \int_1^\infty \frac{\xi}{\xi + z} \, \frac{\log z \, dz}{z},$$

which is asymptotically equal to $\frac{1}{2}(\log \xi)^2$ as $\xi \to \infty$.

Remark 1.5. Numerical simulations for $\psi(\xi) = \xi^{\alpha/2}$ (using the results of [11]) strongly suggest that the error in (3) is in fact of order $O(\frac{1}{n^2})$. There is no numerical evidence for more general functions ψ. We believe that at least when $\psi(\xi) = \sqrt{\xi}$, one can use a method applied in a somewhat similar problem in [10] to obtain a version of (3) with an additional term in the asymptotic expansion and an improved bound of the error term, but this is far beyond the scope of the present article.