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Koshliakov kernel and identities involving the Riemann zeta function

Atul Dixit [a,*], Nicolas Robles [b], Arindam Roy [c], Alexandru Zaharescu [c,d]

[a] *Department of Mathematics, Tulane University, New Orleans, LA 70118, USA*
[b] *Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland*
[c] *Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, USA*
[d] *Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, Bucharest RO-70700, Romania*

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ABSTRACT

Some integral identities involving the Riemann zeta function and functions reciprocal in a kernel involving the Bessel functions $J_z(x)$, $Y_z(x)$ and $K_z(x)$ are studied. Interesting special cases of these identities are derived, one of which is connected to a well-known transformation due to Ramanujan, and Guinand.

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1. Introduction

In their long memoir [19, p. 158, Equation (2.516)], Hardy and Littlewood obtain, subject to certain assumptions unproved as of yet (for example, the Riemann Hypothesis), an interesting modular-type transformation involving infinite series of Möbius function as suggested to them by some work of Ramanujan. By a modular-type transformation, we mean a transformation of the form $F(\alpha) = F(\beta)$ for $\alpha\beta =$ constant. On pages 159–160, they also give a generalization of the transformation for any pair of functions reciprocal to each other in the Fourier cosine transform as indicated to them by Ramanujan.

Let $\Xi(t)$ be Riemann's Ξ-function defined by [30, p. 16]

$$\Xi(t) := \xi(\tfrac{1}{2} + it), \tag{1.1}$$

where

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{1.2}$$

* Corresponding author.
E-mail addresses: adixit@tulane.edu (A. Dixit), nicolas.robles@math.uzh.ch (N. Robles), roy22@illinois.edu (A. Roy), zaharesc@illinois.edu (A. Zaharescu).

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is the Riemann ξ-function [30, p. 16]. Here $\Gamma(s)$ is the gamma function [1, p. 255] and $\zeta(s)$ is the Riemann zeta function [30, p. 1].

A natural way to obtain similar such modular-type transformations is by evaluating integrals of the type

$$\int_0^\infty f(t)\Xi(t)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt,$$

where $f(t)=\phi(it)\phi(-it)$ for some analytic function ϕ, since they are invariant under $\alpha\to 1/\alpha$, although the aforementioned transformation involving series of Möbius function is not obtainable this way. Ramanujan studied an interesting integral of this type in [28].

Motivated by Ramanujan's generalization, the authors of the present paper, in [11], studied integrals of the above type but with the cosine function replaced by a general function $Z\left(\frac{1}{2}+it\right)$, which is an even function of t, real for real t, and depends on the functions reciprocal in the Fourier cosine transform. Several integral evaluations such as the one connected with the general theta transformation formula [9, Equation (4.1)], and those of Hardy [18, Equation (2)] and Ferrar [9, p. 170] were obtained in [11] as special cases by evaluating these general integrals for specific choices of f.

Ramanujan [28] also studied integrals of the form

$$\int_0^\infty f\left(\frac{t}{2},z\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\cos\left(\frac{1}{2}t\log\alpha\right)\,dt,$$

where

$$f(t,z)=\phi(it,z)\phi(-it,z), \tag{1.3}$$

with ϕ being analytic in the complex variable z and in the real variable t. With f being of the form just discussed, in the present paper, we study a generalization of the above integral of the form

$$\int_0^\infty f\left(\frac{t}{2},z\right)\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)Z\left(\frac{1+it}{2},\frac{z}{2}\right)\,dt, \tag{1.4}$$

where the function $Z\left(\frac{1}{2}+it,z\right)$ depends on a pair of functions which are reciprocal to each other in the kernel

$$\cos\left(\frac{\pi z}{2}\right)M_z(4\sqrt{x}\,)-\sin\left(\frac{\pi z}{2}\right)J_z(4\sqrt{x}\,), \tag{1.5}$$

where

$$M_z(x):=\frac{2}{\pi}K_z(x)-Y_z(x).$$

Here $J_z(x)$ and $Y_z(x)$ are Bessel functions of the first and second kinds respectively, and $K_z(x)$ is the modified Bessel function.

We call this kernel the *Koshliakov kernel* since Koshliakov [22, Equation 8] was the first mathematician to construct a function self-reciprocal in this kernel, namely, he showed that for real z satisfying $-\frac{1}{2}<z<\frac{1}{2}$,

$$\int_0^\infty K_z(t)\left(\cos(\pi z)M_{2z}(2\sqrt{xt}\,)-\sin(\pi z)J_{2z}(2\sqrt{xt}\,)\right)\,dt=K_z(x). \tag{1.6}$$