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Periodic averaging principle in quantum calculus

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ABSTRACT

In this paper, we prove a periodic averaging principle for quantum difference equations and present some examples to illustrate our result.

1. Introduction

Consider the ordinary differential equation

$$\begin{cases} x'(t) = \varepsilon f(x(t), t) + \varepsilon^2 g(x(t), t, \varepsilon), \\ x(t_0) = x_0, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter. Assume that f is T-periodic with respect to the second variable. Then, the classical periodic averaging theorem ensures that we can obtain a good approximation of this initial-value problem by neglecting the ε^2-term and taking the average of f with respect to t. More precisely, the solutions of system (1.1) can be approached by the solutions of the averaged system given by

$$\begin{cases} y'(t) = \varepsilon f_0(y(t)), \\ y(t_0) = x_0, \end{cases} \tag{1.2}$$

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where

$$f_0(y) = \frac{1}{T} \int_{t_0}^{t_0+T} f(y,t)dt.$$

The equation (1.1) appears in several problems, such as celestial mechanics, three-body problems, pertubation theory, among others. This type of problem started to be investigated in 18th century with the works by Clairaut, Lagrange and Laplace and since then, it has attracted the attention by several researchers. For more details, see [23].

Therefore, the theory of averaging plays an important rôle in applications, since it can be used to study nonlinear differential equations, perturbation theory, control theory, stability of solutions, bifurcation, among others. This theory has been attracting the attention of many researchers, and the interest in the subject is still growing. See, for instance, [1,12–16,20–22].

The averaging methods were presented in the literature in order to approach several different nonlinear differential systems, such as ordinary differential equations, functional differential equations, impulsive differential equations, generalized differential equations, among others (see [1,12–16,20–22]). Recently, Federson, Mesquita and Slavík in [13] presented in the literature a version of periodic averaging theorem for functional dynamic equations on time scales, where the considered time scale is periodic, that is, if $t \in \mathbb{T}$, then $t+T \in \mathbb{T}$, where T is the period of the time scale $\mathbb{T}$ and also, $\mu(t) = \mu(t+T)$, for every $t \in \mathbb{T}$. However, there are some important time scales which do not satisfy this periodic property, such as the quantum time scale ($q^{\mathbb{N}_0}$, for $q > 1$), which is the basis for the quantum calculus and presents several applications to quantum physics.

On the other hand, quantum calculus (q-calculus) has recently been attracting the attention of many researchers, since it is a powerful tool for applications in several fields of physics, such as cosmic strings and black holes, conformal quantum mechanics, nuclear and high energy physics, fractional quantum Hall effect, and high-T_c superconductors. Thermostatistics of q-bosons and q-fermions can be established using basic numbers and employing the q-calculus based on the Jackson derivative. See, for instance, [17–19,24] and the references therein.

Furthermore, there are several applications of quantum calculus in the study of thermodynamic functions, such as entropy, pressure, internal energy and specific heat. The theory of q-deformed thermostatistics can be more consistently formulated by Jackson derivatives instead of the standard ordinary derivatives of thermostatistics. We mention, for instance, that the q-deformation affects the energy spectrum in a fundamental manner, which can describe complex ensembles or real gases. On the other hand, the standard thermodynamics is usually restricted to describe only ideal gases. Therefore, the physical meaning of q-deformation can be better understood in terms of the Jackson derivative, which corresponds to q-difference equations, than in terms of continuous derivatives and continuous differential equations. For more details, see [17–19,24].

Due to the importance of this quantum calculus and taking into account that the periodic properties of the solutions of q-difference equations are important in order to better understand several physics phenomena, Bohner and Chieochan [7–9] introduced recently (2012) in the literature the following concept of periodicity for functions defined on the quantum time scale (see Definition 1.1 below).

Definition 1.1. A function $f : q^{\mathbb{N}_0} \to \mathbb{R}^n$ is called ω-periodic, where $\omega \in \mathbb{N}$, if

$$q^{\omega} x(q^{\omega} t) = x(t) \quad \text{for all} \quad t \in q^{\mathbb{N}_0},$$

where $q^{\mathbb{N}_0} = \{q^n : \ n \in \mathbb{N}_0\}$.

Using this concept, our goal in this present paper is to prove a periodic averaging principle for quantum calculus. As explained before, due to the lack of a periodicity concept, such result in the setting of q-difference has not been proved in the literature until now.