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Borwein–Preiss variational principle revisited

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ABSTRACT

In this article, we refine and slightly strengthen the metric space version of the Borwein–Preiss variational principle due to Li and Shi (2000) [12], clarify the assumptions and conclusions of their Theorem 1 as well as Theorem 2.5.2 in Borwein and Zhu (2005) [4] and streamline the proofs. Our main result, Theorem 3 is formulated in the metric space setting. When reduced to Banach spaces (Corollary 9), it extends and strengthens the smooth variational principle established in Borwein and Preiss (1987) [3] along several directions.

1. Introduction

The celebrated *Ekeland variational principle* [7] has been around for more than four decades. It almost immediately became one of the main tools in optimization theory and various branches of analysis. The number of publications containing "Ekeland variational principle" in their title has exceeded 200. Several other variational principles followed: due to Stegall [15], Borwein–Preiss [3], Deville–Godefroy–Zizler [5] and others.

Given an "almost minimal" point of a function, a variational principle guaranties the existence of another point and a suitably perturbed function for which this point is (strictly) minimal and provides estimates of the (generalized) distance between the points and also the size of the perturbation. Typically variational principles assume the underlying space to be complete metric (quasi-metric) or Banach and the function (sometimes vector- or set-valued) to possess a kind of semicontinuity.

The principles differ mainly in terms of the class of perturbations they allow. The perturbation guaranteed by the original Ekeland variational principle (valid in general complete metric spaces) is nonsmooth even if the underlying space is a smooth Banach space and the function is everywhere Fréchet differentiable.

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In contrast, the *Borwein–Preiss variational principle* (originally formulated in the Banach space setting) works with a special class of perturbations determined by the norm; when the space is *smooth* (i.e., the norm is Fréchet differentiable away from the origin), the perturbations are smooth too. Because of that, the Borwein–Preiss variational principle is referred to in [3] as the *smooth variational principle*. It has found numerous applications and paved the way for a number of other smooth principles including the one due to Deville–Godefroy–Zizler [5].

The statement of the next theorem mostly follows that of [4, Theorem 2.5.3].

Theorem 1 *(Borwein–Preiss variational principle). Let $(X, \|\cdot\|)$ be a Banach space and function $f : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous. Suppose that $\epsilon > 0$, $\lambda > 0$ and $p \geq 1$. If $x_0 \in X$ satisfies*

$$f(x_0) < \inf_X f + \epsilon, \tag{1}$$

then there exist a point $\bar{x} \in X$ and sequences $\{x_i\}_{i=1}^{\infty} \subset X$ and $\{\delta_i\}_{i=0}^{\infty} \subset \mathbb{R}_+ \setminus \{0\}$ such that $x_i \to \bar{x}$ as $i \to \infty$, $\sum_{i=0}^{\infty} \delta_i = 1$, and

(i) $\|\bar{x} - x_i\| \leq \lambda$ $(i = 0, 1, \ldots)$;

(ii) $f(\bar{x}) + \dfrac{\epsilon}{\lambda^p} \sum_{i=0}^{\infty} \delta_i \|\bar{x} - x_i\|^p \leq f(x_0)$;

(iii) $f(x) + \dfrac{\epsilon}{\lambda^p} \sum_{i=0}^{\infty} \delta_i \|x - x_i\|^p > f(\bar{x}) + \dfrac{\epsilon}{\lambda^p} \sum_{i=0}^{\infty} \delta_i \|\bar{x} - x_i\|^p$ *for all* $x \in X \setminus \{\bar{x}\}$.

When X is a smooth space and $p > 1$, the perturbation functions involved in (ii) and (iii) of the above theorem are smooth.

Among the known extensions of the Borwein–Preiss variational principle, we mention the work by Li and Shi [12, Theorem 1], where the principle was extended to metric spaces (of course at the expense of losing the smoothness) by replacing $\|\cdot\|^p$ in (ii) and (iii) by a more general "gauge-type" function $\rho : X \times X \to \mathbb{R}$. They also strengthened Theorem 1 by showing the existence of $\bar{x}$ and $\{x_i\}_{i=1}^{\infty}$ validating the appropriately adjusted conclusions of the theorem for any sequence $\{\delta_i\}_{i=0}^{\infty} \subset \mathbb{R}_+$ with $\delta_0 > 0$. This last advancement allowed the authors to cover the Ekeland variational principle which corresponds to setting $\delta_i = 0$ for $i = 1, 2, \ldots$ The result by Li and Shi was later adapted in Theorem 2.5.2 in the book by Borwein and Zhu [4].

Another important advancement was made by Loewen and Wang [13, Theorem 2.2] who constructed in the Banach space setting a special class of perturbations subsuming those used in Theorem 1 and established strong minimality in the analogue of the condition (iii) above; cf. [13, Definition 2.1]. Bednarczuk and Zagrodny [2] extended recently the Borwein–Preiss variational principle to vector-valued functions.

In this article which follows the ideas of [3,12,4], we refine and slightly strengthen the metric space version of the Borwein–Preiss variational principle due to Li and Shi [12], clarify the assumptions and conclusions of [12, Theorem 1] and [4, Theorem 2.5.2] and streamline the proofs. When reduced to Banach spaces (Corollary 9), our main result extends and strengthens Theorem 1 along several directions.

1) The assumption $p \geq 1$ for the power index in (ii) and (iii) is relaxed to just $p > 0$. Of course, if $p < 1$, then the perturbation function involved in (ii) and (iii) is not convex.

2) The strict inequality (1) is replaced by the corresponding nonstrict one:

$$f(x_0) \leq \inf_X f + \epsilon.$$

Note that δ_0 must satisfy