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On uniform approximations to hypersingular finite-part integrals ☆

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ABSTRACT

In this paper, new uniform approximation schemes for computation of hypersingular finite-part integrals are studied. The methods are verified to be supremely qualified for oscillatory integrands. In addition, based on the results on Chebyshev approximations, the uniform convergence and computational error bounds are considered. Preliminary numerical results show the stability and efficiency of the schemes.

1. Introduction

Evaluation of the hypersingular finite-part integral

$$I_m(f;c) = \fint_{-1}^{1} \frac{f(x)}{(x-c)^{m+1}} e^{i\omega x} dx, \quad c \in (-1,1), \tag{1.1}$$

can be transferred into the mth-derivative of the Cauchy principle value integral [14,43]

$$I_m(f;c) = \frac{1}{m!} \frac{d^m}{dc^m} \fint_{-1}^{1} \frac{f(x)}{x-c} e^{i\omega x} dx, \tag{1.2}$$

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where f and its derivatives $f^{(j)}(j \le m-1)$ are assumed to be continuous on $[-1, 1]$, $f^{(m)}$ is Hölder continuous [14,43], $i^2 = -1$ and ω is a parameter. For $m = 0$, the integral is the usual Cauchy principle value integral or finite Hilbert transform. For $m = 1$, the integral is the Hadamard finite-part integral [28,43,49,50].

Integrals of the form (1.1) appear frequently in boundary element methods (BEMs) and other numerical computations [3,4,31,43,51], particularly in the area of applied mechanics [31,43]. The efficiency of BEMs often depends upon the efficiency of numerical evaluation of such hypersingular integrals.

In the case $\omega = 0$, the computation of integral (1.1) has been extensively studied, such as the Gauss-type method [14,16,17,23,32,33,41,42], the (composite) Newton–Cotes method [21,39,49–51], and others [12,23, 27,37].

A typical procedure for approximating (1.1) for $\omega = 0$ starts by subtracting out the singularity at c as follows [14,16,17,23,41]:

$$I_m(f; c) = \int_{-1}^{1} \frac{f(x) - \sum_{j=0}^{m} \frac{f^{(j)}(c)}{j!}(x-c)^j}{(x-c)^{m+1}} dx + \sum_{j=0}^{m} \frac{f^{(j)}(c)}{j!} \,⨎_{-1}^{1} \frac{dx}{(x-c)^{m+1-j}}, \tag{1.3}$$

followed by applying some ordinary quadrature rules, such as Gaussian [32,33], Fejér or Clenshaw–Curtis rule [9,10,37] to the first integral on the right-hand side in (1.3). While $⨍_{-1}^{1} \frac{dx}{x-c} = \log \frac{1-c}{1+c}$ and

$$⨎_{-1}^{1} \frac{dx}{(x-c)^{m+1-j}} = \frac{1}{(m-j)!} \frac{d^{m-j}}{dc^{m-j}} ⨍_{-1}^{1} \frac{dx}{x-c}, \quad j = 1, \ldots, m,$$

can be computed exactly.

Although these schemes are simple in general, there are severe numerical cancellations if one of the nodes happens to be close to c [28,29]. Then Gori and Santi [24,25] used the Gauss–Turán quadrature for the Gori–Michelli weight functions to avoid this problem. Milovanović and Spalević [41] generalized the results from [24,25] by using the Chakalov–Popoviciu quadrature. These developed methods can get high accuracy [41], but they involve computation of the higher derivatives of f at quadrature nodes and are dependent on c.

More recently, Sun and Wu [49,50] developed (composite) Newton–Cotes rules for the computation of Hadamard finite-part integrals, and showed the superconvergence. However, these methods are possible to suffer from computational efforts or instability [16,53] as the nodes increase.

Efficiently uniform approximation algorithms are proposed by Hasegawa and Torii [28,29] for the finite Hilbert transform and Hadamard finite-part integral

$$⨍_{-1}^{1} \frac{f(x)}{x-c} dx, \qquad ⨎_{-1}^{1} \frac{f(x)}{(x-c)^2} dx,$$

respectively, based on Chebyshev approximations to $f(x)$. Let

$$p_N(x) = \sum_{k=0}^{N}{}'' a_k^N T_k(x)$$

be the Chebyshev approximate polynomial of $f(x)$ at the Clenshaw–Curtis points $\cos(j\pi/N)$ $(0 \le j \le N)$, where $T_k(x)$ is the Chebyshev polynomial of the first kind given by $T_k(x) = \cos(k\theta)$ and $x = \cos(\theta)$, and