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A note on partial sharing of values of meromorphic functions with their shifts ☆

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ABSTRACT

We introduce the notion of partial value sharing as follows. Let $\overline{E}(a, f)$ be the set of zeros of $f(z) - a(z)$, where each zero is counted only once and a is a meromorphic function, small with respect to f. A meromorphic function f is said to share a partially with a meromorphic function g if $\overline{E}(a, f) \subseteq \overline{E}(a, g)$. We show that partial value sharing of $f(z)$ and $f(z + c)$ involving 3 or 4 values combined with an appropriate deficiency assumption is enough to guarantee that $f(z) \equiv f(z + c)$, provided that $f(z)$ is a meromorphic function of hyper-order strictly less than one and $c \in \mathbb{C}$.

1. Introduction

The famous five point theorem due to R. Nevanlinna [12] says that if two non-constant meromorphic functions f and g share five distinct values ignoring multiplicities (IM), then $f(z) \equiv g(z)$. The beauty of this result lies in the fact that there is no counterpart of this result in the real function theory. Another famous result in this direction is the four point theorem by R. Nevanlinna [12] which states that if two distinct non-constant meromorphic functions f and g share four distinct values counting multiplicities (CM), then $f = T \circ g$, where T is a Möbius map. These results initiated the study of uniqueness of meromorphic functions. The study becomes more interesting if the function g is related to f. Investigations began with the sharing of values by f and f' [5]. Korhonen and Halburd [6,7] and, independently, Chiang and Feng [2,3] developed a parallel difference version to the usual Nevanlinna theory for meromorphic functions of finite order and this gives the direction to study the uniqueness of $f(z)$ and its shift $f(z + c)$, where $c \in \mathbb{C} \setminus \{0\}$.

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Key results of this difference version of Nevanlinna theory were extended by Halburd, Korhonen and Tohge [8] to meromorphic functions of hyper-order strictly less than one.

Throughout this paper, we only consider such meromorphic functions which are non-constant and meromorphic in the whole of $\mathbb{C}$. For such a meromorphic function f, a meromorphic function ω such that $T(r,\omega) = o(T(r,f))$, where $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure, is called a small function of f. The family of all small functions of f is denoted by $\mathcal{S}(f)$ and $\hat{\mathcal{S}}(f) = \mathcal{S}(f) \cup \{\infty\}$.

Recently J. Heittokangas et al. [11,10] considered the problem of value sharing for shifts of meromorphic functions. Precisely, they proved the following results:

Theorem 1.1. *(See [10, Theorem 2.1(a)].) Let f be a meromorphic function of finite order and let $c \in \mathbb{C}\setminus\{0\}$. If $f(z)$ and $f(z+c)$ share three distinct periodic functions $a_1, a_2, a_3 \in \hat{\mathcal{S}}(f)$ with period c CM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

Theorem 1.1 is improved by replacing "sharing three small functions CM" by "2 CM + 1 IM" as:

Theorem 1.2. *(See [11, Theorem 2].) Let f be a meromorphic function of finite order, let $c \in \mathbb{C} \setminus \{0\}$, and let $a_1, a_2, a_3 \in \hat{\mathcal{S}}(f)$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_1, a_2 CM and a_3 IM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

The cases "1 CM + 2 IM" and "3 IM" are left open in [11]. In the present paper, we make certain investigations about the open situations "1 CM + 2 IM" and "3 IM". For basic terms and notations of the value distribution theory of Nevanlinna one may refer to [9,1].

Example 1.3. Consider $f(z) = \frac{(1+\cos z)^2(1-\cos z)}{e^{2iz}}$. Then, we have the following observations:

(i) For $c = \pi$, $f(z)$ and $f(z+c)$ share $0, -1$ IM and ∞ CM.
(ii) $\delta(a, f) = 0$ for all $a \in \mathbb{C}$.
(iii) $\delta(\infty, f) = 1$.

Proofs of (i) and (iii) are immediate. We prove (ii) here. After some simple calculations, one can easily deduce that $f(z) = R(z) \circ e^{iz}$, where

$$R(z) = \frac{z^6 + 2z^5 - z^4 - 4z^3 - z^2 + 2z + 1}{-8z^5}.$$

The only poles of $R(z)$ are at 0 (with multiplicity 5) and at ∞ (with multiplicity 1). Thus, for each $a \in \mathbb{C}$, $R^{-1}(a)$ contains six nonzero complex numbers counted according to their multiplicity. Let $a \in \mathbb{C}$. Let $R^{-1}(a) = \{b_k : k = 1,2,3,4,5,6\}$ (each $b_k \in \mathbb{C} \setminus \{0\}$). Then,

$$N\left(r, \frac{1}{f-a}\right) = \sum_{k=1}^{6} \overline{N}\left(r, \frac{1}{e^{iz} - b_k}\right) = 6T(r, e^z).$$

Thus $N\left(r, \frac{1}{f-a}\right)$ is same for every $a \in \mathbb{C}$. Thus, if a is a deficient value, that is $\delta(a,f) > 0$, then each value in $\mathbb{C}$ is a deficient value which contradicts the fact that set of deficient values of any nonconstant meromorphic function is at most countable. Thus $\delta(a,f) = 0$ for all $a \in \mathbb{C}$.

Definition 1.4. We say that a meromorphic function f shares $a \in \hat{\mathcal{S}}(f)$ partially with a meromorphic function g if $\overline{E}(a,f) \subseteq \overline{E}(a,g)$, where $\overline{E}(a,f)$ is the set of zeros of $f(z) - a(z)$, where each zero is counted only once.