J. Math. Anal. Appl. 435 (2016) 1249–1261

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Coulson-type integral formulas for the general Laplacian-energy-like invariant of graphs I ☆

Lu Qiao, Shenggui Zhang *, Bo Ning, Jing Li

Department of Applied Mathematics, School of Science, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, PR China

A R T I C L E   I N F O

Article history:
Received 13 October 2014
Available online 23 October 2015
Submitted by P. Koskela

Keywords:
Laplacian matrix
Coulson integral formula
General Laplacian-energy-like invariant

A B S T R A C T

Let G be a simple graph. Its energy is defined as $E(G) = \sum_{k=1}^{n} |\lambda_k|$, where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of G. A well-known result on the energy of graphs is the Coulson integral formula which gives a relationship between the energy and the characteristic polynomial of graphs. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ be the Laplacian eigenvalues of G. The general Laplacian-energy-like invariant of G, denoted by $LEL_\alpha(G)$, is defined as $\sum_{\mu_k \neq 0} \mu_k^\alpha$ when $\mu_1 \neq 0$, and 0 when $\mu_1 = 0$, where α is a real number. In this paper we give a Coulson-type integral formula for the general Laplacian-energy-like invariant for $\alpha = 1/p$ with $p \in \mathbb{Z}^+ \backslash \{1\}$. This implies integral formulas for the Laplacian-energy-like invariant, the normalized incidence energy and the Laplacian incidence energy of graphs.

1. Introduction

All graphs considered in this paper are finite and simple. For terminology and notation not defined here, we refer the reader to Cvetković et al. [4].

Let G be a graph with n vertices and m edges. The eigenvalues of the adjacency matrix $A(G)$ of G are said to be the *eigenvalues* of G and form the *spectrum* of G. We denote the eigenvalues of G by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ in non-increasing order. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G, where $D(G) = diag(d_1, d_2, \dots, d_n)$ is the diagonal matrix of vertex degrees of G. It is well known that $L(G)$ is a positive semi-definite symmetric matrix, and moreover 0 is the smallest eigenvalue of $L(G)$. We denote the eigenvalues of $L(G)$ by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$, which are called the *Laplacian eigenvalues* of G.

☆ Supported by NSFC (Nos. 11271300, 11201374 and 11571135), the Doctorate Foundation of Northwestern Polytechnical University (cx201326) and the Fundamental Research Funds for the Central Universities (3102014JCQ01073).

* Corresponding author.
E-mail addresses: water6qiao@mail.nwpu.edu.cn (L. Qiao), sgzhang@nwpu.edu.cn (S. Zhang), ningbo_math84@mail.nwpu.edu.cn (B. Ning), jingli@nwpu.edu.cn (J. Li).

http://dx.doi.org/10.1016/j.jmaa.2015.10.055
0022-247X/

The *energy* of a graph G is defined as $E(G) = \sum_{k=1}^{n} |\lambda_k|$, which is derived from the total π-electron energy [13]. Graph energy has been studied extensively by many mathematicians and chemists, and there have been many results obtained on this invariant of graphs (see [8]). In the theory of graph energy there is an important result called the *Coulson integral formula* which makes it possible to calculate the energy of a graph without knowing its spectrum. For a graph G, its Coulson integral formula is

$$E(G) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \left[n - \frac{ix\phi'_A(G, ix)}{\phi_A(G, ix)} \right] dx,$$

where $\phi_A(G, x)$ is the characteristic polynomial of $A(G)$ (called the *characteristic polynomial* of G). This formula was obtained by Coulson [2], and has many applications in the theory of graph energy (see [8]).

For a graph G, since $\mu_k \geq 0$ for $k = 1, 2, \ldots, n$, it would be trivial to define its Laplacian energy as $\sum_{k=1}^{n} |\mu_k| = \sum_{k=1}^{n} \mu_k = 2m$. Gutman and Zhou [6] defined the *Laplacian energy* of a graph G as

$$LE(G) = \sum_{k=1}^{n} \left| \mu_k - \frac{2m}{n} \right|.$$

Later, Liu and Liu [9] introduced the *Laplacian-energy-like invariant* of G, which is similar to the definition of the graph energy, as

$$LEL(G) = \sum_{k=1}^{n} \sqrt{\mu_k}.$$

This invariant has many similar properties as the energy of a graph. For more results on the Laplacian-energy-like invariant, we refer the reader to the references [7,9,12,16].

In [14], Zhou studied the sum of powers of the Laplacian eigenvalues of graphs, which can be regarded as a generalization of the Laplacian-energy-like invariant. Here we call this invariant the *general Laplacian-energy-like invariant* of graphs.

Definition 1. Let G be a graph of order n, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ the Laplacian eigenvalues of G and α a real number. The general Laplacian-energy-like invariant of G, denoted by $LEL_\alpha(G)$, is defined as $\sum_{\mu_k \neq 0} \mu_k^\alpha$ when $\mu_1 \neq 0$, and 0 when $\mu_1 = 0$.

Obviously, $LEL(G) = LEL_{\frac{1}{2}}(G)$.

In this paper, we obtain a Coulson-type integral formula for the general Laplacian-energy-like invariant of graphs in Section 3. Before that, in Section 2, we give some preliminaries. In Section 4, we present a Coulson-type integral formula for the general energy of polynomials, which is an extension of the general Laplacian-energy-like invariant of graphs, and show that it implies two known integral formulas for the normalized incidence energy and the Laplacian incidence energy.

2. Preliminaries

We first introduce some basic concepts and results from complex analysis which will be used later. Let D be a bounded domain. The boundary of D is denoted by ∂D.

The following two results in complex analysis are well known (see [5]).