J. Math. Anal. Appl. 435 (2016) 1273–1295

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A sharp upper bound for the Hausdorff dimension of the set of exceptional points for the strong density theorem

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ARTICLE INFO

Article history:
Received 21 July 2015
Available online 10 November 2015
Submitted by B.S. Thomson

Dedicated to Professor Athanasios Tsarpalias on the occasion of his retirement

Keywords:
Hausdorff dimension
Besicovitch–Taylor index
Saks' strong density theorem

ABSTRACT

Given an $E \subseteq \mathbb{R}^m$, Lebesgue measurable, we construct a real function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ (depending on E) increasing, with $\lim_{t \to 0^+} \psi(t) = 0$ such that

$$\lim_{\substack{x \in R \\ d(R) \to 0}} \frac{|R \cap E^c|}{|R| \cdot \psi(d(R))} = 0 \qquad \text{for a.e. } x \in E$$

(where R is an interval in $\mathbb{R}^m$ and d stands for the diameter). This gives a new constructive proof of a problem posed by S.J. Taylor (1959) [7, p. 314]. Furthermore, the constructive method we use, gives a sharp upper bound for the Hausdorff dimension of the set of exceptional points, for the strong density theorem of Saks.

1. Introduction

S. Ulam in [5, Problem 146, p. 228] (see also [10, p. 78]) posed the following problem: Suppose that E is any Lebesgue measurable set on the real line. Does there exist a real function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ (depending on E) increasing such that $\lim_{t \to 0^+} \psi(t) = 0$ and

$$\lim_{\substack{x \in I \\ |I| \to 0}} \frac{|I \cap E^c|}{|I| \cdot \psi(|I|)} = 0 \qquad \text{a.e. in } E\,?$$

(Where I denotes an interval in $\mathbb{R}$, E^c is the complement of E and $|\cdot|$ stands for the Lebesgue measure.)

The affirmative answer to this question was given by S.J. Taylor in [7] and of course this is a strengthening of the Lebesgue density theorem. However, the problem, whether does there exist a function ψ independent

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http://dx.doi.org/10.1016/j.jmaa.2015.11.017

of E, has a negative solution (see [7, Theorem 4, p. 312]). Also in [7], the problem for a similar strengthening of the strong density theorem of Saks was posed (for Saks' strong density theorem, see [6, p. 129]).

Problem. (See [7, p. 314].) Given an m-dimensional Lebesgue measurable set E, does there exist a real function $\psi(t)$ increasing with $\lim_{t\to 0^+}\psi(t)=0$ such that

$$\lim_{\substack{x\in R\\ d(R)\to 0}} \frac{|R\cap E^c|}{|R|\cdot\psi(d(R))}=0 \qquad \text{for a.e. } x\in E\,?$$

(R is an interval in $\mathbb{R}^m$, i.e. $R:=I_1\times\cdots\times I_m$, where $I_1,\dots,I_m$ are intervals in $\mathbb{R}$ and d is the diameter of R.)

As S.J. Taylor remarked, the direct methods in [7] did not give this strongest form of the density theorem in m-space.

The affirmative answer to this problem was given by S.J. Taylor in [8], assuming the usual form of the strong density theorem and applying an ingenious strengthening of Egoroff's theorem. The answer given is not constructive and gives no information on how the function ψ depends on the set E.

In the present paper we give a new constructive answer to the problem of S.J. Taylor in [7], that gives a function ψ not only for a particular set, but for an uncountable class of sets.

To be precise, given a sequence $\{(w_n,u_n)\colon n\in\mathbb{N}\}$ in $\mathbb{R}^2$ with $w_n>0$, $u_n>0$ for $n\in\mathbb{N}$ such that the series $\sum_{n=1}^{\infty} w_n\cdot u_n$ converges, we say that a set $E\subseteq\mathbb{R}^2$ *belongs to* $\{(w_n,u_n)\colon n\in\mathbb{N}\}$ if $E=\bigcup_{n=1}^{\infty}(I_n\times J_n)$, where $I_n\times J_n$ are disjoint intervals in $\mathbb{R}^2$ with $|I_n|=w_n$, $|J_n|=u_n$ for $n\in\mathbb{N}$. Clearly, every bounded open set belongs to some sequence as above and there are uncountably many sets belonging to a given sequence (this kind of definition in one dimension is also considered in [2] for a different purpose).

As result (see Theorem 4.9), given a sequence $\{(w_n,u_n)\colon n\in\mathbb{N}\}$ as above, we construct a $\psi:\mathbb{R}^+\to\mathbb{R}^+$ (depending on $\{(w_n,u_n)\colon n\in\mathbb{N}\}$) increasing, with $\lim_{t\to 0^+}\psi(t)=0$ such that for every set E belonging to $\{(w_n,u_n)\colon n\in\mathbb{N}\}$

$$\lim_{t\to 0}\frac{\sup\limits_{\substack{x\in A\times B\\ diam(A\times B)<t}}\dfrac{|(A\times B)\cap E^c|}{|A\times B|}}{\psi(t)}=0 \qquad \text{for a.e. } x\in E$$

(where $A\times B$ is an interval in $\mathbb{R}^2$).

(We remark that the Problem in [7, p. 314] is somewhat misstated, since the ratio following the limit is not uniquely determined, for given $d(A\times B)$.)

It should be noted that, this kind of result can be obtained in one dimension by the direct methods of [7]. Also, our method equally works in any dimension, but we restrict to $\mathbb{R}^2$ for notational simplicity.

Furthermore, the constructive method we use, gives a sharp upper bound of the Hausdorff dimension of the set of exceptional points, for the strong density theorem of Saks.

To be precise, in [3] A.S. Besicovitch proved that, given a perfect set $E\subset[0,1]$ and denoting by φ the sequence $a_1,a_2,\dots$ of the lengths of interior complementary intervals of E, the Hausdorff dimension of the set of exceptional points, for the Lebesgue density theorem (for E), is bounded above by the Besicovitch–Taylor index (or exponent of convergence) of the sequence $\{a_n:\ n\in\mathbb{N}\}$ (defined as

$$e_{BT}\{a_n:\ n\in\mathbb{N}\}:=\inf\{c>0:\sum_{n=1}^{\infty}a_n^c \text{ converges}\} \qquad \text{(see [9, p. 34]))}$$

and this bound is sharp.