J. Math. Anal. Appl. 435 (2016) 1296–1320

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

General monotonicity and interpolation of operators

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ARTICLE INFO

Article history:
Received 6 February 2015
Available online 23 October 2015
Submitted by Richard M. Aron

Keywords:
General monotonicity
Fourier coefficients
$L(p,q)$ spaces
Interpolation of operators

ABSTRACT

Using interpolation properties of cones of general monotone functions, we prove the equivalence of the $L(p,q)$ norms of such functions and their Fourier transforms.

1. Introduction

Given a class X, we shall denote by X^+ the family of positive elements in X.

Throughout the article we use the following weighted L^q and l^q quasi-norms:

Definition 1.1. Let f be a measurable function on $\mathbb{R}^+ = (0,\infty)$ and let $\{a_n\}$ be a sequence of complex numbers. For $0 < p \le \infty$ and $0 < q \le \infty$ define:

$$\|f\|_{L^q_{w(p,q)}} = \left\| f(x) \cdot x^{\frac{1}{p}-\frac{1}{q}} \right\|_{L^q}; \quad \|\{a_n\}\|_{l^q_{w(p,q)}} = \left\| \left\{ a_n \cdot n^{\frac{1}{p}-\frac{1}{q}} \right\} \right\|_{l^q}. \tag{1}$$

To simplify the language, we will refer to the quantities (1) as norms.

$L^q_{w(p,q)}$ and $l^q_{w(p,q)}$ are the spaces of such functions and sequences for which the corresponding norms are finite.

For any measurable function f on an arbitrary measure space (Ω, Σ, μ), so that $\mu\{|f| > \gamma\} < \infty$ for all $\gamma > 0$, we define its decreasing rearrangement, f^*, on $(0,\infty)$, so that $\lambda\{f^* > \gamma\} = \mu\{|f| > \gamma\}$ for all $\gamma > 0$, where λ is Lebesgue measure on the line. We define similarly the rearrangement of a sequence $\{a_n\}$, and denote it by $\{a_n^*\}$.

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http://dx.doi.org/10.1016/j.jmaa.2015.09.087
0022-247X/

Recall the definition of the Lorentz spaces:

Definition 1.2. Let f and $\{a_n\}$ be such that f^* and $\{a_n^*\}$ exist. For $0 < p < \infty$ and $0 < q \leq \infty$, or $p = q = \infty$, define

$$\|f\|_{L(p,q)} = \|f\|_{L(p,q)(\Omega,\Sigma,\mu)} = \|f^*\|_{L^q_{w(p,q)}}\ ;\ \|\{a_k\}\|_{l(p,q)} = \|\{a_k^*\}\|_{l^q_{w(p,q)}}\,. \tag{2}$$

$L\left(p,q\right)$ and $l\left(p,q\right)$ are the spaces of such functions and sequences for which the corresponding norms are finite. $L\left(p,q\right)$ and $l\left(p,q\right)$ are called Lorentz spaces.

For any pair of positive functions, Q_1 and Q_2, let us write $Q_1 \sim Q_2$ if there exists a constant $C > 0$ so that $\frac{1}{C}Q_1 \leq Q_2 \leq CQ_1$.

Definition 1.3. Given a sequence $\{a_n\}$, the function $f\left(x\right) = a_{\lceil x \rceil}$ is called its associated function.

Lemma 1.4. *Let f be the function associated with $\{a_n\}$, $0 < p < \infty$, $0 < q \leq \infty$, or $p = q = \infty$. Then*

$$\|f\|_{L^q_{w(p,q)}} \sim \|\{a_k\}\|_{l^q_{w(p,q)}}\,. \tag{3}$$

Also, f^* exists if and only if $\{a_n^*\}$ exists and if they do then

$$\|f\|_{L(p,q)} \sim \|\{a_k\}\|_{l(p,q)}\,. \tag{4}$$

Proof. For $q < \infty$:

$$\|f\|_{L^q_{w(p,q)}} = \left(\sum_{k=1}^{\infty}\int\limits_{k-1}^{k} x^{\frac{q}{p}-1}\left|f\left(x\right)\right|^q dx\right)^{\frac{1}{q}}$$
$$= \left(\sum_{k=1}^{\infty}|a_k|^q\int\limits_{k-1}^{k} x^{\frac{q}{p}-1}dx\right)^{\frac{1}{q}} \sim \left(\sum_{k=1}^{\infty}|a_k|^q\,k^{\frac{q}{p}-1}\right)^{\frac{1}{q}} = \|\{a_k\}\|_{l^q_{w(p,q)}}\,.$$

For $q = \infty$:

$$\sup_{k-1\leq x<k} x^{\frac{1}{p}}\left|f\left(x\right)\right| = k^{\frac{1}{p}}\left|a_k\right| \Longrightarrow \|f\|_{L^\infty_{w(p,\infty)}} = \|\{a_k\}\|_{l^\infty_{w(p,\infty)}}$$

proving (3). (4) is proved similarly. □

G.H. Hardy and J.E. Littlewood showed that there is a norm equivalence between a function and the sequence of its Fourier coefficients provided that either the function or the sequence is nonnegative and decreasing:

Theorem 1.5. *(See G.H. Hardy and J.E. Littlewood [7].) Assume that $\{c_n\} \searrow 0$, $f(x) = \sum_{n=0}^{\infty} c_n \cos nx$ or $f(x) = \sum_{n=1}^{\infty} c_n \sin nx$. Then for all $p \in (1,\infty)$,*

$$\|f\|_{L^p(0,\pi)} \sim \|\{c_n\}\|_{l(p',p)}\,.$$