J. Math. Anal. Appl. 435 (2016) 1321–1334

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Semiattractors of set-valued semiflows

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A R T I C L E   I N F O

Article history:
Received 15 October 2015
Available online 12 November 2015
Submitted by H. Frankowska

Keywords:
Semiattractor
Semifractal
Set-valued semiflow
Iterated function system
Random dynamical system
Semigroup of Markov operators

A B S T R A C T

We consider smallest closed invariant subsets of the phase space with respect to arbitrary set-valued semiflows. Such sets, called semiattractors, attract all trajectories of singletons but not necessary compact or bounded subsets. In particular, there are systems admitting semiattractors which do not have global attractors as usually understood. We show some sufficient conditions on the existence of such a set. We also prove lower semicontinuous dependence of semiattractors for strict semiflows of l.s.c. multifunction and the existence of semigroups of Markov operators generated by such semiflows. We are motivated by asymptotic properties of semiflows of multifunctions connected with iterated function systems as well as random dynamical systems.

1. Introduction

In [18] J.E. Hutchinson proved the following famous result. Consider a complete metric space (X, ϱ) and a finite family of transformations $S_i : X \to X$, $i \in \{1, \dots, N\}$. If all of those transformations are contractions, i.e. for every $i \in \{1, \dots, N\}$ there is a constant $L_i \in (0, 1)$ such that $\varrho(S_i(x), S_i(y)) \le L_i \varrho(x, y)$ for $x, y \in X$ (such a system is called hyperbolic) then there exists a unique compact set $\mathcal{A} \subset X$ such that it is a fixed point of the operator $F_* : \mathcal{K}(X) \to \mathcal{K}(X)$ defined by

$$F_*(K) := \mathrm{cl} \bigcup_{i=1}^{N} S_i(K) \quad \text{for} \quad K \in \mathcal{K}(X),$$

where $\mathcal{K}(X)$ denotes a hyperspace of all non-empty compact subsets of X. Moreover, $h(F_*^n(K), \mathcal{A}) \to 0$ as $n \to \infty$ for every $K \in \mathcal{K}(X)$, where F_*^n stands for n-th iterate of the operator F_* and h denotes the usual Hausdorff metric in the hyperspace $\mathcal{K}(X)$. It is interesting that exactly *this* set $\mathcal{A}$ is called the attractor of an iterated function system $\{S_i :\ i \in \{1, \dots, N\}\}$. Observe that in the considered case the above convergence

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http://dx.doi.org/10.1016/j.jmaa.2015.11.029
0022-247X/

in the Hausdorff metric is equivalent to the existence of topological (Kuratowski's) limit $\mathcal{A} = \lim_n \ F^n(K)$, where F^n are iterates of a multifunction F defined by $F(x) := \{S_i(x) : \ i \in \{1, \ldots, N\}\}$, $x \in X$.

If transformations S_i are randomly chosen with some probability (dependent or not on x) for a given x_0 one can obtain a homogeneous Markov chain by the equation $x_{n+1} = S_{i_n}(x_n)$ for $n \in \mathbb{N}_0$, where now (i_n) is a sequence of independent random variables on I. If iterated function system is hyperbolic this Markov chain converges to a stationary Borel probability measure μ_* on X, i.e. $prob(x_n \in A) \to \mu_*(A)$ as $n \to \infty$ for every Borel subset $A \subset X$ and the attractor $\mathcal{A}$ of the system is a support of the measure μ_*.

In [23] A. Lasota and J. Myjak noticed that in some computer applications when probabilistic algorithms with iterated function systems are used the considered systems are not hyperbolic and do not have a compact attractor, although the associated Markov chains tends to some unique density μ_*. Those iterated function systems are mostly such that there are few transformations consisting a hyperbolic subsystem. The question arises: what is a deterministic set associated with asymptotic behavior of the operator F_* (or the multifunction F) which coincides with support of μ_*? The solution leads to the new interesting class of subsets called semiattractors or semifractals of iterated function systems (see [23–25], also [14]). Such sets need not to be compact, but are smallest closed non-empty sets invariant with respect to the Barnsley–Hutchinson operator F_* and attracts trajectories of all singletons from X.

Developing the ideas by A. Lasota and J. Myjak we noticed in [14] and [16] that any cocycle mapping induces a net of multifunctions (called state multifunctions) and we associated with this net its attractor as well as semiattractor in the similar way as in [25]. In particular, we have shown [16, Theorem 6.2, Theorem 6.6] that if a cocycle has a structure of random dynamical system the semiattractor coincides with a minimal global point attractor. It is known that random dynamical system can have a point attractor instead of the usual desired global attractor (cf. [10–12]).

In this paper we deal with semiattractors of arbitrary semiflows of multifunctions. We show (see Example 5.2 below) that for cocycles the net of state multifunctions forms a semiflow and if we are going to study white noise random dynamical systems the induced semiflow of state multifunctions is strict. In the theory of asymptotic behavior of set-valued dynamical systems mostly studied are attracting sets of upper semi-continuous mappings. Since the state multifunction of iterated function systems as well as cocycle mappings with continuous members are lower semi-continuous it seams to be reasonable discuss semiattractors of semiflows of multifunction possessing this type of continuity. Moreover, from the probabilistic point of view such multifunctions are strictly connected with semigroups of Markov operators (see Section 7 below). Then the most important property of subsets of the state space X with respect to the net of these multifunctions is positive subinvariance (see [26] and [29]). We accent this property of semiattractors (instead of usually desired negative subinvariance of attractors, see [2,27]). In the whole paper the language of topological limits is consequently used as a main tool instead of common Hausdorff metric. This point of view seems to be simpler, more general and, in consequence, fruitful.

The paper is organized as follows: Sections 2–4 contain some needed definition and facts concerning topological limits, lower semicontinuous multifunctions and both iterated function systems and random dynamical systems, respectively. Section 5 is devoted to the definition and properties of semiattractors of general as well as lower semicontinuous set-valued semiflows. Presented here results generalize those of [25] where discrete semiflows generated by iterative process of single multifunctions were considered. The 6-th section contains the theorem on lower semicontinuous dependence of semiattractors on parameters. We also prove some corollaries for iterated function systems as well as random dynamical systems. We hope that presented results could be applied to the very new branch of the theory of control systems i.e. random dynamical systems with inputs and outputs, where indexed families of random dynamical systems appear naturally (see [13]). In section 7 some probabilistic approach is presented. We prove that every strict and measurable semiflow of closed-valued l.s.c multifunctions induces some Markov–Feller process described by a semigroup of transition Markov operators acting on the space of finite Borel measures. It is shown what is the relationship between support of an attractive invariant measure of this Markov semigroup and the