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Uniqueness of complete maximal surfaces in certain Lorentzian product spacetimes

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ABSTRACT

We considered complete maximal surfaces in a Lorentzian manifold given by the product of the negative definite real line and a 2-dimensional Riemannian manifold, such that the Gauss curvature of the Riemannian fiber is bounded from below. The main purpose of this work is to characterize the surfaces satisfying a comparison involving the height function and the shape operator as slices. In order to obtain the results, we developed a proper extension of a classical result by Nishikawa, for the Ricci bounds depending on the distance function. Finally, we present non-trivial examples of surfaces to emphasize the necessity of the assumptions we required in our results.

1. Introduction

In [3,4], A.L. Albujer and L.J. Alías established Calabi–Bernstein results for complete maximal surfaces in a Lorentzian product spacetime $-\mathbb{R}\times M^2$. In particular, when the Riemannian surface M^2 has non-negative Gauss curvature, they proved that any complete maximal surface must be totally geodesic. Besides, if M^2 is non-flat, the authors concluded that it must be a slice $\{t\} \times M^2$. The necessity of the assumption on the Gauss curvature can be observed from the examples of maximal surfaces in $-\mathbb{R} \times \mathbb{H}^2$, where $\mathbb{H}^2$ is the hyperbolic plane, constructed in [1]. In [11], G. Li and I. Salavessa generalized such results of [3] to higher dimension and codimension.

The first author and H.F. de Lima exhibit in [12] an example of a (non-totally geodesic) complete spacelike surface of constant mean curvature (CMC) in $-\mathbb{R} \times \mathbb{H}^2$ such that the hyperbolic angle function is constant. In [8] the second author, M. Caballero and M. Rubio worked in 3-dimensional Generalized Robertson–Walker spacetimes considering maximal surfaces with uniqueness results for the case the fiber has non-negative Gauss curvature generalizing results of Albujer and Alías in [3].

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The main aim of this work is to present new Calabi–Bernstein properties of complete maximal surfaces in a Lorentzian product spacetime where the Gauss curvature of the fiber M^2 satisfies $K_M \geq -\kappa$ for some $\kappa \in \mathbb{R}$, $\kappa > 0$.

For the case $-\mathbb{R} \times \mathbb{H}^2$, it is known [3] that there are complete maximal surfaces which are not totally geodesic. Thus, it naturally arises the question to decide what additional assumptions are needed to conclude that a complete maximal surface in $-\mathbb{R} \times M^2$, where $K_M \geq -\kappa$, must be totally geodesic.

Our technique is based on a proper extension of a well-known result by Nishikawa in [13] and relies within the applications of the generalized maximum principle due to Yau [15] on complete Riemannian manifolds. In fact, we previously proved an extension of [13, Lemma 2] to the case the Ricci curvature Ric is no longer bounded by a constant but by a more general function $G(r)$ of the distance r from a fixed point on the manifold (Lemma 4.1):

Let M^n be an $n(\geq 2)$-dimensional complete Riemannian manifold such that $\mathrm{Ric} \geq -G(r)$ for a function G such that

$$G(0) \geq 1, \quad G' \geq 0 \ \text{ and } \ G^{-1/2} \notin L^1[0, \infty).$$

If u is a non-negative function on M^n satisfying

$$\Delta u \geq \beta\, u^2, \text{ for a constant } \beta > 0, \tag{4.1}$$

then $u = 0$.

Using this technical result, we arrive at the main goal of this paper (Theorem 4.2):

Let $\overline{M}^3 = -\mathbb{R} \times M^2$ be a Lorentzian product spacetime, such that the Gauss curvature K_M of its Riemannian fiber M^2 satisfies $K_M \geq -\kappa$, for some positive constant κ. Let Σ be a complete maximal surface in $\overline{M}^3$ such that its Gauss–Kronecker curvature satisfies $K_G \leq G(r)$. If the height function h and the shape operator A of Σ satisfy

$$|\nabla h|^2 \leq \frac{|A|^2}{\kappa}, \tag{4.5}$$

then Σ must be a slice.

This theorem uses three main assumptions: maximality, the inequality (4.5) and the controlled growth of K_G. From its proof we observe that the condition on the growth of K_G can be replaced by the same condition on the growth of the norm of gradient of the height function (Corollary 4.6):

Let $\overline{M}^3 = -\mathbb{R} \times M^2$ be a Lorentzian product spacetime, such that the Gauss curvature K_M of its Riemannian fiber M^2 satisfies $K_M \geq -\kappa$, for some positive constant κ. Let Σ be a complete maximal surface in $\overline{M}^3$ such that the hyperbolic angle between the pointing future unit normal vector field N and ∂_t is $G(r)$, and additionally inequality (4.5) holds then Σ must be a slice.

In order to show that the assumptions in Theorem 4.2 cannot be dropped, we will analyze in Section 6 three examples of spacetime graphs in $-\mathbb{R} \times \mathbb{H}^2$, for $\mathbb{H}^2$ given by the Poincaré half-plane model of half plane. Example 6.1, [12], $u(x, y) = a \ln y$ with $|a| < 1$, shows that maximality cannot be removed. In fact, we cannot even replace it by constant mean curvature. Considering the previously quoted result we see that Example 6.2, [1], $u(x, y) = a \ln(x^2 + y^2)$ for $a < \frac{1}{2}$, lacks only the hypothesis of the inequality (4.5). Finally