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Non-singular solutions of a Rayleigh–Plesset equation under a periodic pressure field ☆

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A B S T R A C T

In this paper we prove the existence of infinitely many non-singular subharmonic solutions, as well as the presence of complex dynamics, for the equation with singularity

$$\ddot{u} + \frac{g_1}{u^\beta} - \frac{g_2}{u^\gamma} = h_0(t)u^\delta,$$

where $h_0(t)$ is a T-periodic stepwise function. Our result is stable with respect to small perturbations and can be applied to the Rayleigh–Plesset equation governing the dynamics of a bubble immersed in an infinite domain of liquid.

1. Introduction and motivation

This paper is devoted to the study of periodic non-singular solutions for the second order equation with singularities

$$\ddot{u} + c\frac{\dot{u}}{u^\alpha} + \frac{g_1}{u^\beta} - \frac{g_2}{u^\gamma} = h_0(t)u^\delta, \tag{1.1}$$

where c, g_1, g_2, α, β and γ are positive constants with $\beta > \gamma$. The function $h_0 : \mathbb{R} \to \mathbb{R}$ is a T-periodic function with $h_0 \in L^\infty(\mathbb{R})$. We wish to find periodic (both harmonic and subharmonic) solutions to (1.1) which are *nonsingular*, that is with $u(t) > 0$ for all $t \in \mathbb{R}$.

The study of (1.1) is motivated by the famous Rayleigh–Plesset equation for bubble dynamics which is an important tool to investigate phenomena related to cavitation. In fluid mechanics, the Rayleigh–Plesset

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equation is an ordinary differential equation concerning the dynamics of a spherical bubble of radius $R(t)$ (where t is the time) immersed in an infinite domain of liquid whose temperature and pressure far from the bubble are T_∞ and $p_\infty(t)$, respectively. The temperature T_∞ is assumed to be constant, while $p_\infty(t)$, is assumed to be a known (and perhaps controlled) input which regulates the growth or collapse of the bubble [1,3]. A first form of this equation was introduced by Lord Rayleigh in 1917 neglecting surface tension and liquid viscosity and keeping the pressure p_∞ constant. A more general form of the equation including effects not considered by Rayleigh, was proposed in 1949 by M.S. Plesset (see [16]). A typical form of the Rayleigh–Plesset equation is the following

$$\frac{p_B - p_\infty(t)}{\rho_L} = R\frac{d^2R}{dt^2} + \frac{3}{2}\left(\frac{dR}{dt}\right)^2, \tag{1.2}$$

where $R = R(t)$ is the bubble radius at time t, p_B is the pressure within the bubble at the bubble wall and ρ_L is the density of the surrounding liquid, assumed to be constant. Different variants of (1.2) have been proposed, depending on the choice of the pressure term p_B which may depend on R and also on t. In this respect, two important forms of (1.2) are the following

$$R\ddot{R} + \frac{3}{2}(\dot{R})^2 = \frac{1}{\rho_L}\left(p_i(t) - p_\infty(t) - \frac{2\sigma}{R} - \frac{4\mu}{R}\dot{R}\right) \tag{1.3}$$

and

$$R\ddot{R} + \frac{3}{2}(\dot{R})^2 = \frac{1}{\rho_L}\left(p_{i,eq}\left(\frac{R_0}{R}\right)^{3\kappa} - p_\infty(t) - \frac{2\sigma}{R} - \frac{4\mu}{R}\dot{R}\right) \tag{1.4}$$

(cf. [16]). In (1.3) the surface-tension constant and the coefficient of the liquid viscosity are σ and μ, respectively, while p_i denotes the pressure inside the bubble, usually taken as a constant. This is the typical model considered for a spherical bubble of vapor. The second form of the equation arises from the study of a bubble in which the medium filling the cavity is a non-condensable gas. It is also supposed that the gas follows a polytropic law of compression of the form $p_i = p_{i,eq}\left(\frac{R_0}{R}\right)^{3\kappa}$, with polytropic exponent κ. The coefficient $p_{i,eq}$ is a positive constant and $R_0 > 0$ denotes an equilibrium radius (unstable) around which oscillations of bubble radius take place. In [16] Plesset and Prosperetti described some pioneering works on small-amplitude forced radial oscillations which arise when a bubble is immersed in a sound field of wavelength large compared with the bubble radius. Such a sound field is introduced by assuming

$$p_\infty(t) = P_\infty(1 + \varepsilon\cos(\omega t)),$$

where P_∞ is the average ambient pressure, ω is the sound frequency, and ε is the dimensionless amplitude of the pressure perturbation which is supposed to be small. Under the assumption of polytropic behavior and with the neglect of thermal and acoustic dissipation, equation (1.4) takes the form of

$$R\ddot{R} + \frac{3}{2}(\dot{R})^2 = \frac{1}{\rho_L}\left(p_{i,eq}\left(\frac{R_0}{R}\right)^{3\kappa} - P_\infty(1 - \eta\cos(\omega t)) - \frac{2\sigma}{R} - \frac{4\mu}{R}\dot{R}\right), \tag{1.5}$$

where, in general, the dimensionless pressure amplitude η is not necessarily small. Quoting Plesset and Prosperetti (1977): *The numerical studies that can be found in the literature have shown the extreme richness of this equation which appears to lie beyond the capabilities of the available analytical techniques* [16]. As we shall see later, as result of our investigation, we are able to prove in a rigorous analytical manner the presence of complex dynamics for a variant of (1.5) in which we consider as $p_\infty(t)$ a stepwise function.