J. Math. Anal. Appl. 435 (2016) 1410–1425

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Structure of solutions set of nonlinear eigenvalue problems ☆

Sun Jingxian, Xu Xian *

Department of Mathematics, Jiangsu Normal University, Xuzhou, Jiangsu, 221116, PR China

A R T I C L E   I N F O

Article history:
Received 16 September 2013
Available online 12 November 2015
Submitted by P.J. McKenna

Keywords:
Bifurcation theories
Leray–Schauder degree
Connected component

A B S T R A C T

In this paper, by using the global bifurcation theory we obtain some results for structure of the solution set of some nonlinear equations with parameters. We show a result concerning the existence of a connected component, which either has a loop, or is unbounded both from left and right hand side. Especially, in this paper we also give some sufficient conditions for a bounded connected component of solution set being a loop.

1. Introduction

Bifurcation phenomena arise in many fields of mathematical physics. The studying of their nature is of practical as well as theoretical importance. There have been many results concerning the local or global bifurcation theories by using the topological degree theories, general set point theories and the linearization considerations; See [1–3,5–10,12–18,20,21]. In particular, some authors obtained the results concerning the existence of the connected component which has a loop. Now let us recall some such kind of global bifurcations results; See [2,7,9,15]. The author of [9] considered the problem of global bifurcation of nontrivial solutions of

$$x = \lambda Lx + H(x, \lambda),$$

where L is linear and compact on a Banach space E, $H : E \times \mathbb{R} \to E$ is completely continuous, $H(0, \lambda) = 0$ on $\mathbb{R}$ and $\|x\|^{-1}\|H(x, \lambda)\| \to 0$ as $\|x\| \to 0$ locally uniformly in λ. The author of [9] proved global bifurcation of two semi-global continua $C_\mu^\pm$ of non-trivial solutions bifurcating from $(0, \mu)$, where μ is a characteristic value of L. It is shown by the author that if μ has geometric multiplicity 1 and odd algebraic multiplicity, then C_μ^+ and C_μ^- are both unbounded, or $C_\mu^+ \cap C_\mu^- \setminus \{(0, \mu)\} \neq \emptyset$. Moreover, if C_μ^+ and C_μ^- are both bounded, $C_\mu^+ \cap C_\mu^-$ contains $(0, \alpha)$, where $\alpha \neq \mu$, that is, C^+ and C^- have a loop structure.

☆ This paper is supported by NSFC11501260, Natural Science Foundation of Jiangsu Education Committee (09KJB110008) and Qing Lan Project.

* Corresponding author.
E-mail address: xuxian68@163.com (X. Xu).

http://dx.doi.org/10.1016/j.jmaa.2015.11.024

The authors of [15] studied bifurcations of positive solutions from the trivial branch of one parameter families of compact vector fields on ordered Banach spaces. They proved the existence of a connected component of positive solutions emanating from a singular point λ_0 of the linearization which is either unbounded or goes back to the trivial branch, provided: (i) the kernel of the linearization of the family at λ_0 is one dimensional and generated by a vector in the positive cone, (ii) the Leray–Schauder degree changes at λ_0, (iii) the derivative of the compact part sends positive elements into the interior of the cone. Obviously, as the connected component emanating from a singular point λ_0 goes back to the trivial branch, then the connected component may form a loop structure.

Now an interesting question is: whether there is a nonlinear operator equation with parameters which has a bifurcation diagram that is an upright counterpart of those of the main theorems in [2,7,9,15]. The purpose of this paper is to study the structure of solution set of some nonlinear operator equations. We shall show a result for the existence of a connected component C, which either has a loop, or is unbounded both from left and right hands. Roughly speaking, we may think the bifurcation diagrams of our main results – Theorems 2.1 and 2.2 – are an upright counterpart of those of the main theorems in [2,7,9,15]. In this paper we shall give some sufficient conditions for a bounded connected component of solutions set being a loop; See Corollaries 2.1–2.3 and Theorem 3.1 below. These results can be applied to elliptic boundary value problems, ordinary differential boundary value problems etc. to give the existence results for loops emitting from non-trivial solutions as well as trivial solutions. From our points of view, although we can obtain the existence results for loops by using the main results of [2,7,9,15], however, those results of [2,7,9,15] would be more suitable for finding loops emitting from trivial solution. This seems to be a difference between our main results and those of [2,7,9,15].

2. The main results

First let us introduce some symbols we will use in the sequel of this paper. Let M be a metric space, $U \subset M$. In the sequel we will use $\overline{U}^M$ to denote the closure of U in M, $\partial_M U$ to denote the boundary of U in M. Let E be a real Banach space. For simplicity we will use $\overline{U}$ to denote the closure of U in the Banach space E, and ∂U the boundary of U in the Banach space E.

In the sequel of this paper we always assume that $A : E \to E$ is a completely continuous operator,

$$L = \overline{\{(\lambda, x) \in \mathbb{R} \times E : x = \lambda Ax, x \neq 0\}}^{\mathbb{R}\times E},$$

and

$$S = \{(\lambda, x) \in \mathbb{R} \times E : x = \lambda Ax\}.$$

From [20] we have the following Definitions 2.1 and 2.2.

Definition 2.1. Let C be a connected component of L, $(\lambda^*, x^*) \in C$. Then $(\lambda^*, x^*) \in C$ is called a regular point of C, if there exists $r_0 > 0$ small enough such that $(\{\lambda^*\} \times B(0, r_0)) \cap L = \{(\lambda^*, x^*)\}$, where $B(0, r_0) = \{x \in E : \|x\| < r_0\}$.

Assume that (λ^*, x^*) is a regular point on a connected component C of L. For each $0 < r < r_0$, let

$$O^+((\lambda^*, x^*), r) = \{(\lambda, x) : (\lambda, x) \in \mathbb{R} \times E, (|\lambda - \lambda^*|^2 + \|x - x^*\|^2)^{\frac{1}{2}} < r, \lambda > \lambda^*\},$$
$$O^-((\lambda^*, x^*), r) = \{(\lambda, x) : (\lambda, x) \in \mathbb{R} \times E, (|\lambda - \lambda^*|^2 + \|x - x^*\|^2)^{\frac{1}{2}} < r, \lambda < \lambda^*\}.$$