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Composition operators induced by injective homomorphisms on infinite weighted graphs

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ABSTRACT

We deal with composition operators induced by injective homomorphisms on infinite weighted graphs from a viewpoint of reproducing kernel Hilbert space theory. The representation formula for their adjoint operators is given in the terms of the frames constructed from reproducing kernels, and de Branges–Rovnyak spaces induced by composition operators are studied.

1. Introduction

Let G be a graph with at most countably many vertices. The vertex set of G will be denoted by $V = V(G)$ and the edge set by $E = E(G)$. Let $\deg(x)$ denote the number of edges connected at a vertex x. We assume that $\deg(x)$ is finite for each x in V, and these graphs are said to be locally finite. Further, all graphs appearing in this paper are assumed to be non-directed, have no loops, have a vertex called the origin denoted as 0_G, and are connected, that is, there exists a finite path from x to y for any x and y in V. Let $W_{x,y}$ denote a real-valued function on $V \times V$ such that

$$W_{x,y} > 0 \quad (\{x, y\} \in E), \quad W_{x,y} = 0 \quad (\{x, y\} \notin E) \quad \text{and} \quad W_{x,y} = W_{y,x}.$$

Then graph G equipped with $W_{x,y}$ is called a weighted graph or a network.

Definition 1.1. Let $G_1 = (V_1, E_1, W^{(G_1)})$ and $G_2 = (V_2, E_2, W^{(G_2)})$ be weighted graphs. A mapping φ from V_1 to V_2 is called a homomorphism from G_1 to G_2 if

$$W^{(G_1)}_{x,y} \le W^{(G_2)}_{\varphi(x),\varphi(y)} \quad (x, y \in V_1).$$

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In this paper, this inequality will be abbreviated as $W_{x,y} \le W_{\varphi(x),\varphi(y)}$ if no confusion occurs. Further, if homomorphism φ is a bijective mapping from V_1 to V_2 satisfying

$$W^{(G_1)}_{x,y} = W^{(G_2)}_{\varphi(x),\varphi(y)} \quad (x, y \in V_1)$$

then φ will be called an isomorphism from G_1 to G_2.

Throughout this paper, we will deal only with injective homomorphisms which preserve origins, that is, we assume the following two conditions:

(i) $\varphi(x) \ne \varphi(y)$ if $x \ne y$,
(ii) $\varphi(0_{G_1}) = 0_{G_2}$.

The purpose of this paper is to study the composition operators induced by injective homomorphisms using reproducing kernel Hilbert space methods. This paper is organized as follows. In Section 2, we show how one can construct from a weighted graph a reproducing kernel Hilbert space of real-valued functions with a natural frame (in the sense of wavelet theory). In Section 3, basic properties of composition operators induced by injective homomorphisms are shown. In particular, the representation formula for their adjoint operators is given in the terms of the frames constructed from reproducing kernels. As suggested in [5], de Branges–Rovnyak space theory would be a suitable framework for dealing with injective homomorphisms on graphs in functional analysis. In Section 4, de Branges–Rovnyak spaces induced by injective homomorphisms on infinite weighted graphs are studied.

2. Hilbert space $\mathcal{H}_G$

$\mathcal{E}(\cdot,\cdot)$ will denote the weighted discrete Dirichlet form on $V \times V$ defined as follows:

$$\mathcal{E}(u, v) = \frac{1}{2} \sum_{x,y \in V} W_{x,y}(u(x) - u(y))(v(x) - v(y)),$$

where u and v are real-valued functions on V. It is easy to see that $\mathcal{E}(u, u) = 0$ if and only if u is constant, because G is connected.

Definition 2.1. Let $\mathcal{H}_G$ denote the real Hilbert space consisting of real-valued functions on V such that $u(0_G) = 0$ and $\mathcal{E}(u, u)$ is finite, that is, we set

$$\mathcal{H}_G = \{u : \mathcal{E}(u, u) < +\infty \quad \text{and} \quad u(0_G) = 0\} \quad \text{and} \quad \|u\|^2_{\mathcal{H}_G} = \mathcal{E}(u, u).$$

Let δ_x denote the delta function at x, and we set $W_x = \sum_{\{x,y\} \in E} W_{x,y}$. It is readily seen that

$$\mathcal{E}(\delta_x, \delta_y) = \begin{cases} W_x & (x = y) \\ -W_{x,y} & (x \ne y), \end{cases} \quad (x, y \in V),$$

and thus the set $\{\delta_x : x \in V \setminus \{0_G\}\}$ is contained in $\mathcal{H}_G$.

Theorem 2.1. *For any x in V, there exists a unique function k_x in $\mathcal{H}_G$ such that $\langle u, k_x\rangle_{\mathcal{H}_G} = u(x)$ for any u in $\mathcal{H}_G$, that is, $\mathcal{H}_G$ is a reproducing kernel Hilbert space.*

Proof. Although this fact is well known, we give a proof for the sake of readers. We fix an arbitrary vertex x. Then there exists a finite path $P = \{x_0, x_1, \ldots, x_n\}$ from $x_0 = 0_G$ to $x_n = x$ in G by the assumption that