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General convolution identities for Bernoulli and Euler polynomials

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A B S T R A C T

Using general identities for difference operators, as well as a technique of symbolic computation and tools from probability theory, we derive very general kth order ($k \geq 2$) convolution identities for Bernoulli and Euler polynomials. This is achieved by use of an elementary result on uniformly distributed random variables. These identities depend on k positive real parameters, and as special cases we obtain numerous known and new identities for these polynomials. In particular we show that the well-known identities of Miki and Matiyasevich for Bernoulli numbers are special cases of the same general formula.

1. Introduction

The Bernoulli and Euler numbers and polynomials have been studied extensively over the last two centuries, both for their numerous important applications in number theory, combinatorics, numerical analysis and other areas of pure and applied mathematics, and for their rich structures as interesting objects in their own right. The Bernoulli numbers B_n, $n = 0, 1, 2, \dots$, can be defined by the exponential generating function

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \qquad (|z| < 2\pi). \tag{1.1}$$

They are rational numbers, the first few being $1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \dots$, with $B_{2k+1} = 0$ for $k \geq 1$. For the most important properties see, for instance, [1, Ch. 23] or its successor [19, Ch. 24]. Other good references are [11,14], or [18]. For a general bibliography, see [5].

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Numerous linear and nonlinear recurrence relations for these numbers are known, and such relations also exist for the Bernoulli *polynomials* and for Euler numbers and polynomials which will be defined later. This paper deals with *nonlinear* recurrence relations, the prototype of which is Euler's well-known identity

$$\sum_{j=0}^{n}\binom{n}{j}B_jB_{n-j} = -nB_{n-1}-(n-1)B_n \qquad (n\ge 1). \tag{1.2}$$

This can also be seen as a convolution identity. Two different types of convolution identities were discovered more recently, namely

$$\sum_{j=2}^{n-2}\frac{B_jB_{n-j}}{j(n-j)}-\sum_{j=2}^{n-2}\binom{n}{j}\frac{B_jB_{n-j}}{j(n-j)} = 2H_n\frac{B_n}{n} \qquad (n\ge 4) \tag{1.3}$$

by Miki [17], where $H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$ is the nth harmonic number, and

$$(n+2)\sum_{j=2}^{n-2}B_jB_{n-j}-2\sum_{j=2}^{n-2}\binom{n+2}{j}B_jB_{n-j} = n(n+1)B_n \qquad (n\ge 4) \tag{1.4}$$

by Matiyasevich [16]; see also [2] and the references therein. These two identities, which are remarkable in that they combine two different types of convolutions, were later extended to Bernoulli polynomials by Gessel [10] and by Pan and Sun [20], respectively. Gessel [10] also extended (1.3) to third-order convolutions, i.e., sums of products of three Bernoulli numbers. Later Agoh [2] found different and simpler proofs of the polynomial analogues of (1.3) and (1.4) and proved numerous other similar identities involving Bernoulli, Euler, and Genocchi numbers and polynomials. Subsequently Agoh and the first author [3] extended the polynomial analogue of (1.4) to convolution identities of arbitrary order, and did the same for Euler polynomials. Meanwhile, following different lines of investigation, Dunne and Schubert [6] derived an identity that has both (1.3) and (1.4) as special cases, and Chu [4] obtained a large number of convolution identities, some of them extending (1.3) and (1.4).

It is the purpose of this paper to contribute to the recent work summarized above and to further extend the identities (1.3) and (1.4) of Miki and Matiyasevich. In Section 2 we state a general result concerning second-order convolutions, and derive some consequences. In Section 3 we introduce a symbolic notation with a related calculus, and use it to state and prove a very general identity for Bernoulli polynomials. This is then used in Section 4, along with some methods from probability theory, to prove a general higher-order convolution identity which gives the main result of Section 2 as a special case. In Section 5 we apply most of the methods from Sections 3 and 4 to Euler polynomials and again obtain general higher-order convolution identities. Finally, in Section 6, we state and prove several further consequences of each of our main theorems. We conclude this paper with some further remarks in Section 7.

2. Identities for Bernoulli polynomials

The *Bernoulli polynomials* can be defined by

$$B_n(x) := \sum_{j=0}^{n}\binom{n}{j}B_jx^{n-j}, \tag{2.1}$$

or equivalently by the generating function

$$\frac{ze^{xz}}{e^z-1} = \sum_{n=0}^{\infty}B_n(x)\frac{z^n}{n!} \qquad (|z|<2\pi). \tag{2.2}$$