J. Math. Anal. Appl. 435 (2016) 1525–1551

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Algebraic contraction rate for distance between entropy solutions of scalar conservation laws

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A R T I C L E   I N F O

Article history:
Received 25 April 2014
Available online 12 November 2015
Submitted by K. Nishihara

Keywords:
Gradient flow
Wasserstein distance
Scalar conservation laws
Burgers equation

A B S T R A C T

We establish an algebraic contraction rate in a modified Wasserstein distance for solutions of scalar conservation laws with uniformly convex flux. We also show that our estimate is optimal w.r.t. scaling in time and discuss why it gives non-trivial information in relation to the stability of the rarefaction wave.

Contents

1. Introduction

The aim of this paper is to establish a contraction result in a modified transport distance for entropy solutions to a scalar conservation law

$$\partial_t \theta + \partial_x f(\theta) = 0, \tag{1.1}$$

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http://dx.doi.org/10.1016/j.jmaa.2015.11.027
0022-247X/

with uniformly convex flux f. This estimate gives non-trivial information in relation to the stability of the rarefaction wave solution of (1.1) and its contraction rate turns out to be optimal in terms of scaling in time.

The intuition behind the arguments to derive this contraction result is geometric in nature. As noted already in [7], the Burgers' equation

$$\partial_t \theta + \partial_x(\theta(\theta - 1)) = 0, \tag{1.2}$$

a special case of equation (1.1), can be written formally as a gradient flow of the energy

$$F(\theta) = \int_{\mathbb{R}} x\theta(x)dx,$$

with respect to the two-phase Wasserstein space. In [7] these insights are derived in the physical context of a relaxed version of a model of the flow of two immiscible fluids of different density and mobility in a porous medium. One well-known benefit of formally writing partial differential equations as a gradient flow in Wasserstein space is deriving contraction results, provided the energy is semi-convex.

Unfortunately, this is not the case when writing the Burgers' equation as a gradient flow as above, i.e. the energy F is not semi-convex. This can formally be seen by Lemma 3.2, which tells us that the Hessian of F is given by

$$\text{Hess } F(\theta) = \frac{-\partial_x f'(\theta)}{2} \, \text{id}\,.$$

Since $\eta(t,x) =: \eta_t(x) = H(x)$, where H is the Heaviside function, is a solution to (1.2) with formally

$$\partial_x f'(\eta_t) = 2\partial_x \eta_t = +\infty,$$

we see that in general the Hessian of F is not bounded from below. This is not surprising, since intuitively speaking this corresponds to the non-uniqueness of solutions to the initial value problem related to (1.2), see the discussion in [5]. To reestablish uniqueness for this initial value problem, a well-known selection principle is introduced: the notion of entropy solution. That these entropy solutions also play a special role in the above context of the gradient flow interpretation is the content of [5], namely the fact that the time-discretized gradient flow for F with respect to the two-phase Wasserstein metric converges to the entropy solution.

The idea of our work here is to use special distinguishing features of the entropy solution to obtain a contraction-like estimate for these solutions. Indeed, a careful analysis of the Hessian of F along entropy solutions shows us that the well-known Oleinik condition ensures a kind of semi-convexity which improves over time, namely

$$\text{Hess } F(\theta_t) \geq -\frac{1}{2t} \, \text{id}\,.$$

Geometrically speaking we establish a semi-convexity of the energy landscape along certain trajectories, despite the fact that the global energy landscape of F is highly non-convex. Altogether this provides an interesting example of a gradient flow which satisfies a certain contractivity in spite of the fact that it lives in an energy landscape that is not globally semi-convex.

2. Preliminaries

2.1. Preliminaries on scalar conservation laws

In this section, we quickly recall some facts about the entropy solution to a scalar conservation law, which can be for example found in [10].