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Splittings and cross-sections in topological groups

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ABSTRACT

This paper deals with the splitting of extensions of topological abelian groups. Given topological abelian groups G and H, we say that $\mathrm{Ext}(G, H)$ is trivial if every extension of topological abelian groups of the form $1 \to H \to X \to G \to 1$ splits. We prove that $\mathrm{Ext}(A(Y), K)$ is trivial for any free abelian topological group $A(Y)$ over a zero-dimensional k_ω-space Y and every compact abelian group K. Moreover we show that if K is a compact subgroup of a topological abelian group X such that the quotient group X/K is a zero-dimensional k_ω-space, then there exists a continuous cross section from X/K to X. In the second part of the article we prove that $\mathrm{Ext}(G, H)$ is trivial whenever G is a product of locally precompact abelian groups and H has the form $\mathbb{T}^\alpha \times \mathbb{R}^\beta$ for arbitrary cardinal numbers α and β. An analogous result is true if $G = \prod_{i \in I} G_i$ where each G_i is a dense subgroup of a maximally almost periodic, Čech-complete group for which both $\mathrm{Ext}(G_i, \mathbb{R})$ and $\mathrm{Ext}(G_i, \mathbb{T})$ are trivial.

1. Introduction

In this paper we consider the splitting of extensions of topological abelian groups. An extension of topological abelian groups is a short exact sequence $1 \to H \to X \to G \to 1$, where H, X, G are topological abelian groups and all maps in the sequence are assumed to be continuous and open homomorphisms when considered as maps onto their images. Throughout this paper we will refer to it simply as "an extension". The extension splits if it is equivalent to $1 \to H \to H \times G \to G \to 1$ in the natural sense; this means that H splits as a subgroup of X. The *splitting problem* can be formulated as the problem of finding conditions on H and G under which all such extensions split. If this property holds we will say that $\mathrm{Ext}(G, H)$ is trivial, where $\mathrm{Ext}(G, H)$ stands for the set of equivalence classes of extensions of the form $1 \to H \to X \to G \to 1$.

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Moskowitz [14] studied this problem in the realm of locally compact abelian groups (from now on LCA groups). He proved that if G is a LCA group and $H \cong \mathbb{T}^\alpha \times \mathbb{R}^n$, for some non-negative integer n and an arbitrary cardinal α, then $\mathrm{Ext}(G, H)$ is trivial. Later on Fulp and Griffith established that a LCA group H has the property that $\mathrm{Ext}(G, H)$ is trivial for all connected LCA groups G if and only if $H \cong \mathbb{T}^\alpha \times \mathbb{R}^n$ (Theorem 3.3 in [13]).

Some particular extensions of not necessarily locally compact topological abelian groups were studied by Cabello in [7]. He introduced the concept of a quasi-homomorphism in the category of topological groups. He found that every quasi-homomorphism $q\colon G \to H$ induces an extension which he denotes by $1 \to H \to H \oplus_q G \to G \to 1$, and that every extension of this form splits provided that H is $\mathbb{R}$ or $\mathbb{T}$ and G is a product of locally compact abelian groups. The notion of quasi-homomorphism is based on that of quasi-linear map which was studied by Domański [10] in the framework of topological vector spaces.

Topological vector spaces, when considered in their additive structure, constitute an important class of topological abelian groups for which this theory is fairly well understood, at least in some concrete cases. Namely, there are well-known necessary and sufficient conditions on the spaces E and F under which every extension of Fréchet spaces $0 \to F \to L \to E \to 0$ splits [5,17]. These results have many applications, for instance to problems concerning partial differential or convolution operators.

The paper is organized as follows. We prove in Section 2 that $\mathrm{Ext}(G, K)$ is trivial whenever K is a compact abelian group and G is the free abelian topological group over a zero-dimensional k_ω-space. As a by-product we obtain the following result, which is interesting in itself: If K is a compact subgroup of a topological abelian group X such that the quotient group X/K is a zero-dimensional k_ω-space, then there exists a continuous cross section from X/K to X.

The main result of Section 3 is Theorem 3.13 which states that $\mathrm{Ext}(G, H)$ is trivial whenever $H = \mathbb{T}^\alpha \times \mathbb{R}^\beta$ with α and β arbitrary cardinal numbers and $G = \prod_{i\in I} G_i$ where each G_i is a dense subgroup of a maximally almost periodic, Čech-complete group for which both $\mathrm{Ext}(G_i, \mathbb{R})$ and $\mathrm{Ext}(G_i, \mathbb{T})$ are trivial. An important ingredient in the proof of this result is Theorem 3.5, which establishes that $\mathrm{Ext}(G, M)$ is trivial whenever G is any topological abelian group, M is metrizable and locally compact, and $\mathrm{Ext}(G/P, M)$ is trivial for each P in a cofinal family of admissible subgroups of G.

1.1. Notation, terminology, and preliminary facts

As usual, ω is the set of natural numbers, $\mathbb{R}$ is the set of real numbers, and $\mathbb{C}$ is the set of complex numbers. The unit circle of $\mathbb{C}$ with the topology inherited from $\mathbb{C}$ is denoted by $\mathbb{T}$.

We are mainly interested in abelian groups, although some of our results are valid in a more general setting. If H is a closed subgroup of a topological group G, then G/H is the space of left cosets of H with the quotient topology. This is of course a topological group when H is a normal subgroup of G.

We use multiplicative notation for the group operation. Accordingly, we denote the neutral element of a group G by 1_G or simply by 1 if there is no risk of confusion. Given a topological group G, we will denote by $\mathcal{N}_1(G)$ the family of all neighborhoods of 1 in G.

A topological abelian group G is said to be a MAP (maximally almost periodic) group if the continuous homomorphisms of G to $\mathbb{T}$ separate points of G.

A topological space X is a k_ω-space if it has the weak topology with respect to an increasing sequence of compact subsets whose union is X.

By the *character* (resp. *pseudocharacter*) of a point x in a topological space X we mean the minimum cardinality of a basis of neighborhoods of x in X (resp. of a family of open neighborhoods of x whose intersection is $\{x\}$). Every point of a topological group G has the same (pseudo)character and we refer to it simply as the *(pseudo)character of G*.