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On a measure of noncompactness in the space of functions with tempered increments

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A B S T R A C T

Based on a certain criterion for relative compactness in the space of functions with tempered increments, we construct a formula expressing a set function in that space. We prove that the mentioned function is a sublinear measure of noncompactness in the space in question. Moreover, we show that with help of that measure of noncompactness, we can obtain an existence result for a nonlinear quadratic integral equation of Hammerstein type in the space of function satisfying the Hölder condition.

1. Introduction

The principal aim of the paper is to construct a formula expressing a measure of noncompactness in the Banach spaces of real functions defined on a compact metric space and having increments tempered by a given modulus of continuity. In that construction we will use a certain criterion for relative compactness in the mentioned Banach spaces obtained in [9]. It turns out that the measure of noncompactness constructed in the paper is very useful and handy in applications. We show its applicability in the space of functions satisfying the Hölder condition (Hölder functions, in short) on a given closed and bounded interval. More precisely, we prove that a nonlinear quadratic integral equation of Hammerstein type has a solution in the Hölder function space. In the proof of that result we will apply a fixed point theorem of Darbo type and the technique associated with the measure of noncompactness in question.

The approach based on the use of the measure of noncompactness constructed in the paper turns out to be very convenient and allows us to obtain a more general result in comparison with the application of

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Schauder fixed point principle in conjunction with the criterion for relative compactness mentioned above (cf. [6,9]).

It is worthwhile mentioning that the measure of noncompactness defined in this paper is the first one which is constructed in the space of functions with tempered increments and used in proving existence results in that space. In particular, as far as we know, it is also the first measure of noncompactness defined and applied in the Hölder function space.

2. The space of functions with tempered increments

The content of this section is closely patterned on a suitable material from the paper [9].

Namely, assume that (M, d) is a given metric space. For further purposes we will assume that this space is compact, although the requirement that M is a bounded metric space would be sufficient in our preliminary considerations.

Next, denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, \infty)$. Moreover, we denote by $\mathbb{N}$ the set of positive integers (natural numbers).

Further, a function $\omega(\varepsilon) = \omega : \mathbb{R}_+ \to \mathbb{R}_+$ will be called the modulus of continuity if $\omega(0) = 0$, $\omega(\varepsilon) > 0$ for $\varepsilon > 0$ and ω is nondecreasing on $\mathbb{R}_+$. We will also assume (but not always) that the modulus of continuity $\omega(\varepsilon)$ is continuous at the point $\varepsilon = 0$ i.e., $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now, denote by the symbol $C(M)$ the space of all functions $x : M \to \mathbb{R}$ being continuous on M. The space $C(M)$ will be endowed with the classical supremum norm $||\cdot||_\infty$ what means that for $x \in C(M)$ we put

$$||x||_\infty = \sup \{|x(u)| : u \in M\}.$$

Next, for a given function $x \in C(M)$ and for $\varepsilon > 0$ we define the quantity $\nu(x, \varepsilon)$ by putting

$$\nu(x, \varepsilon) = \sup \{|x(u) - x(v)| : u, v \in M,\ d(u, v) \leqslant \varepsilon\}.$$

The function $\varepsilon \to \nu(x, \varepsilon)$ is called the modulus of continuity of the function x.

In what follows assume, as before, that (M, d) is a compact metric space and $\omega = \omega(\varepsilon)$ is a fixed modulus of continuity. Denote by $C_\omega(M)$ the set of all real functions defined on M such that their increments are tempered by the modulus of continuity ω. This means that a function $x = x(u)$ belongs to $C_\omega(M)$ whenever $x : M \to \mathbb{R}$ and there exists a constant $k_x > 0$ such that

$$|x(u) - x(v)| \leqslant k_x\, \omega\,(d(u, v)) \tag{2.1}$$

for all $u, v \in M$. In other words, $x \in C_\omega(M)$ if and only if the quantity

$$\sup \left\{ \frac{|x(u) - x(v)|}{\omega\,(d(u, v))} : u, v \in M, u \neq v \right\}$$

is finite. Obviously, in view of (2.1) we have that $x \in C_\omega(M)$ if and only if there exists a constant $k_x > 0$ such that $\nu(x, \varepsilon) \leqslant k_x\, \omega(\varepsilon)$ for any $\varepsilon > 0$. Thus, we can say that the space $C_\omega(M)$ consists of all real functions defined on M, such that the moduli of continuity of those functions are tempered by a given modulus of continuity $\omega = \omega(\varepsilon)$.

Further, let us recall that we can define the norm in the space $C_\omega(M)$ in the following way [9]

$$||x|| = |x(u_0)| + \sup \left\{ \frac{|x(u) - x(v)|}{\omega\,(d(u, v))} : u, v \in M, u \neq v \right\}, \tag{2.2}$$

where u_0 is an arbitrarily fixed element of M.