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Approaching the solving of constrained variational inequalities via penalty term-based dynamical systems

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A B S T R A C T

We investigate the existence and uniqueness of (locally) absolutely continuous trajectories of a penalty term-based dynamical system associated to a constrained variational inequality expressed as a monotone inclusion problem. Relying on Lyapunov analysis and on the ergodic continuous version of the celebrated Opial Lemma we prove weak ergodic convergence of the orbits to a solution of the constrained variational inequality under investigation. If one of the operators involved satisfies stronger monotonicity properties, then strong convergence of the trajectories can be shown.

1. Introduction and preliminaries

This paper is motivated by the increasing interest in solving constrained variational inequalities expressed as monotone inclusion problems of the form

$$0 \in Ax + N_C(x), \tag{1}$$

where $\mathcal{H}$ is a real Hilbert space, $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is a maximally monotone operator, $C = \operatorname{argmin} \Psi$ is the set of global minima of the proper, convex and lower semicontinuous function $\Psi : \mathcal{H} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ fulfilling $\min \Psi = 0$ and $N_C : \mathcal{H} \rightrightarrows \mathcal{H}$ is the normal cone of the set $C \subseteq \mathcal{H}$ (see [4–6,8,17,18,26,27]). One can find in the literature iterative schemes based on the forward–backward paradigm for solving (1) (see [5,6,26,27]), that perform in each iteration a proximal step with respect to A and a subgradient step with respect to the penalization function Ψ.

Recently, even more complex structures have been analyzed, like monotone inclusion problems of the form

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$$0 \in Ax + Dx + N_C(x), \tag{2}$$

where $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is a maximally monotone operator, $D : \mathcal{H} \to \mathcal{H}$ is a (single-valued) cocoercive operator and $C \subseteq \mathcal{H}$ is the (nonempty) set of zeros of another cocoercive operator $B : \mathcal{H} \to \mathcal{H}$, see [8,17,18].

In this paper we are concerned with addressing monotone inclusion problem (2) from the perspective of dynamical systems. More precisely, we associate to this constrained variational inequality a first-order dynamical system formulated in terms of the resolvent of the maximal monotone operator A, which has as discrete counterparts penalty-type numerical schemes already considered in the literature in the context of solving (2). Let us mention that dynamical systems of similar implicit type have been investigated in [1,3,9,12,19–21].

In the first part of the manuscript we study the existence and uniqueness of (locally) absolutely continuous trajectories generated by the dynamical system, by appealing to arguments based on the Cauchy–Lipschitz–Picard Theorem (see [25,29]). In the second part of the paper we investigate the convergence of the trajectories to a solution of the constrained variational inequality (2). We use as tools Lyapunov analysis combined with the continuous version of the Opial Lemma. Under the fulfillment of a condition expressed in terms of the Fitzpatrick function of the cocoercive operator B we are able to show ergodic weak convergence of the orbits. Moreover, if the operator A is strongly monotone, we can prove even strong (non-ergodic) convergence for the generated trajectories.

For the reader's convenience we present in the following some notations which are used throughout the paper (see [10,14,28]).

Let $\mathcal{H}$ be a real Hilbert space with *inner product* $\langle\cdot,\cdot\rangle$ and associated *norm* $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. The *normal cone* of $S \subseteq \mathcal{H}$ is defined by $N_S(x) = \{u \in \mathcal{H} : \langle y - x, u\rangle \le 0 \ \forall y \in S\}$, if $x \in S$ and $N_S(x) = \emptyset$ for $x \notin S$. Notice that for $x \in S$, $u \in N_S(x)$ if and only if $\sigma_S(u) = \langle x, u\rangle$, where σ_S is the support function of S, defined by $\sigma_S(u) = \sup_{y\in S}\langle y, u\rangle$.

For an arbitrary set-valued operator $M : \mathcal{H} \rightrightarrows \mathcal{H}$ we denote by $\operatorname{Gr} M = \{(x, u) \in \mathcal{H} \times \mathcal{H} : u \in Mx\}$ its *graph*, by $\operatorname{dom} M = \{x \in \mathcal{H} : Mx \neq \emptyset\}$ its *domain*, by $\operatorname{ran} M = \{u \in \mathcal{H} : \exists x \in \mathcal{H} \text{ s.t. } u \in Mx\}$ its *range* and $M^{-1} : \mathcal{H} \rightrightarrows \mathcal{H}$ its *inverse operator*, defined by $(u, x) \in \operatorname{Gr} M^{-1}$ if and only if $(x, u) \in \operatorname{Gr} M$. We use also the notation $\operatorname{zer} M = \{x \in \mathcal{H} : 0 \in Mx\}$ for the *set of zeros* of the operator M. We say that M is *monotone* if $\langle x - y, u - v\rangle \ge 0$ for all $(x, u), (y, v) \in \operatorname{Gr} M$. A monotone operator M is said to be *maximally monotone*, if there exists no proper monotone extension of the graph of M on $\mathcal{H} \times \mathcal{H}$. Let us mention that in case M is maximally monotone, $\operatorname{zer} M$ is a convex and closed set [10, Proposition 23.39]. We refer to [10, Section 23.4] for conditions ensuring that $\operatorname{zer} M$ is nonempty. If M is maximally monotone, then one has the following characterization for the set of its zeros:

$$z \in \operatorname{zer} M \text{ if and only if } \langle u - z, w\rangle \ge 0 \text{ for all } (u, w) \in \operatorname{Gr} M. \tag{3}$$

The operator M is said to be *γ-strongly monotone* with $\gamma > 0$, if $\langle x - y, u - v\rangle \ge \gamma\|x - y\|^2$ for all $(x, u), (y, v) \in \operatorname{Gr} M$. Notice that if M is maximally monotone and strongly monotone, then $\operatorname{zer} M$ is a singleton, thus nonempty (see [10, Corollary 23.37]).

The *resolvent* of M, $J_M : \mathcal{H} \rightrightarrows \mathcal{H}$, is defined by $J_M = (\operatorname{Id} + M)^{-1}$, where $\operatorname{Id} : \mathcal{H} \to \mathcal{H}$, $\operatorname{Id}(x) = x$ for all $x \in \mathcal{H}$, is the *identity operator* on $\mathcal{H}$. Moreover, if M is maximally monotone, then $J_M : \mathcal{H} \to \mathcal{H}$ is single-valued and maximally monotone (cf. [10, Proposition 23.7 and Corollary 23.10]). We will also use the *Yosida approximation* of the operator M, which is defined by $M_\alpha = \frac{1}{\alpha}(\operatorname{Id} - J_{\alpha M})$, for $\alpha > 0$.

The *Fitzpatrick function* associated to a monotone operator M, defined as

$$\varphi_M : \mathcal{H} \times \mathcal{H} \to \overline{\mathbb{R}},\ \varphi_M(x, u) = \sup_{(y,v)\in \operatorname{Gr} M} \{\langle x, v\rangle + \langle y, u\rangle - \langle y, v\rangle\},$$

is a convex and lower semicontinuous function and it will play an important role throughout the paper. Introduced by Fitzpatrick in [24], this notion opened the gate towards the employment of convex analysis