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A note on a nonlinear elliptic problem with a nonlocal coefficient

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ABSTRACT

In this paper, we investigate a nonlocal and nonlinear elliptic problem,

$$\begin{cases} -a\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda u + u^p \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{P}$$

where $N \le 3$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, a is a nondegenerate continuous function, $p > 1$ and $\lambda \in \mathbb{R}$. We show several effects of the nonlocal coefficient a on the structure of the solution set of (P). We first introduce a scaling observation and describe the solution set by using that of an associated semilinear problem. This allows us to get unbounded continua of solutions (λ, u) of (P). A rich variety of new bifurcation and multiplicity results are observed. We also prove that the nonlocal coefficient can induce up to uncountably many solutions in a convenient way. Lastly, we give some remarks from the variational point of view.

1. Introduction

In this paper, we consider a nonlinear elliptic problem involving the Dirichlet energy,

$$\begin{cases} -a\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda u + u^p \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{P}$$

where $N \le 3$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. In addition, assume a is a continuous function such that $a \ge a_0$ for some constant $a_0 > 0$, $1 < p < \infty$ if $N = 1, 2$, $1 < p \le 5$ if $N = 3$ and $\lambda \in \mathbb{R}$ is a parameter. Our aim is to show the influence of the nonlocal coefficient a on the solvability

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of (P). To do this, we study the structure of the solution set, the bifurcation phenomena and the multiplicity of solutions of (P) in a convenient way.

The nonlocal problems involving the Dirichlet energy are introduced in several stages of natural sciences. In the theory of the nonlinear vibrations, it appears as a wave equation [23]. For earlier mathematical developments, see [8]. Readers may also find recent studies in [9] and the references therein. On the other hand, a parabolic problem is introduced as a model equation for the dynamics of the population density of bacterias and also the heat conduction, see [14] and [15]. In particular, in [15], they indicate that it can admit several equilibria and has the energy structure. This motivates them to investigate the asymptotic behavior of the solution. More recently, the stationary and thus, elliptic problems with nonlinear reaction terms, such as (P), attract much attention [5,12,16,18,20,25,26,28–31,33,36,40]. Using the variational or topological techniques, the authors investigate the existence of solutions. For example, the 3-superlinear at infinity case is considered in [40] and some references therein. That is, they consider (P) with $a(t) = a_0 + \alpha t$ where $a_0, \alpha > 0$ and the nonlinearity $f \in C(\mathbb{R})$ such that

$$\lim_{u\to\infty} \frac{f(u)}{u^3} = \infty.$$

This is a natural assumption in view of the mountain pass type geometry [7]. With the asymptotically linear condition at zero and some additional ones, they get the existence of solutions. On the other hand, Perera–Zhang [36] and Liang–Li–Shi [26] investigate a delicate problem including the asymptotically linearity at zero and the asymptotically 3-linearity at infinity, say,

$$\lim_{u\to\infty} \frac{f(u)}{u^3} = Const.$$

It is worth remarking that this case has a close relation to the nonlinear eigenvalue problem,

$$\begin{cases} -\int_\Omega |\nabla u|^2 dx\ \Delta u = \mu u^3 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Some interesting studies on its eigenvalues and functions are observed in their works. In addition, in [26], they indicate the difficulty caused by the lack of the Ambrosetti–Rabinowitz type condition [7]. Applying the tool in [22] based on the monotonicity trick [39], they get the solvability including a bifurcation result. Notice that our nonlinearities $\lambda u + u^p$ with $p > 3$ and $p = 3$ are typical examples of these two assumptions above respectively. In addition, we deal with the 3-sublinearity at infinity, $1 < p < 3$, which has not been considered yet to our best knowledge, and the critical case, $N = 3$ and $p = 5$. Recently the critical case has been attacked by the author [31]. He solves the problem by utilizing the pioneering argument by Brezis–Nirenberg [10] with the concentration compactness result by Lions [27]. An interesting thing is that, he gets a solution which attains the local minimum of the energy in addition to a mountain pass type solution. See Theorem 5.1 there. Since if Ω is a ball, (P) with $a(t) = 1$ admits at most one positive solution, see [1], we may conclude that this multiplicity is, in fact, induced by the nonlocal coefficient. Now, this reminds us of the earlier works by Chipot et al. in [14] and [15] stated in the beginning of this paragraph. As in them, the author's result implies that the nonlocal coefficient can induce the multiplicity of stationary solutions even for the problem with the nonlinear reaction term. Indeed, we can find a related result in [12], which says that the concave-convex problem [6] may have the third positive solution. Readers can refer to its introduction or Theorem 2.4. Our work is inspired by these results. As noted in the first paragraph, one of the aims of this paper is to show how the nonlocal coefficient can affect the multiplicity of solutions of (P). Actually, we will see that (P) admits a rich variety of multiplicity results by the combined effect of the nonlocal coefficient and the nonlinear reaction term. Before beginning our main argument, we introduce a convenient observation below.