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Uniqueness of large positive solutions for a class of radially symmetric cooperative systems ☆

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ABSTRACT

This paper shows the uniqueness of the large solution of a class of radially symmetric sublinear elliptic systems of cooperative type. This result is based on the strong maximum principle, rather than on the blow-up rates of the large solutions, as it is common in most the available results for single equations.

1. Introduction

This paper studies the existence and uniqueness of the solution of the, eventually singular, elliptic problem

$$\begin{cases} -\Delta u_i = \lambda_i u_i + \sum_{j=1, j\neq i}^{n} a_{ij} u_j - \alpha_i(d(x)) f_i(u_i) u_i & \text{in} \quad \Omega, \\ u_i = M & \text{on} \quad \partial\Omega, \end{cases} \qquad 1 \leq i \leq n, \tag{1.1}$$

where $M \in (0, \infty]$, λ_i, $a_{ij} \in \mathbb{R}$ for all $1 \leq i \leq j \leq n$, $j \neq i$, $n \in \mathbb{N}$, with $a_{ij} > 0$, and

$$d(x) := \operatorname{dist}(x, \partial\Omega), \qquad x \in \Omega,$$

is the distance to the boundary function. The kind of domains Ω considered here are the ball and the annulus. So,

$$\Omega \in \{B_R(x_0), A_{R_1,R_2}(x_0)\}, \tag{1.2}$$

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where $x_0 \in \mathbb{R}^N$, $N \geq 1$, $R > 0$, $R_2 > R_1 > 0$, and

$$B_R(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < R\},$$
$$A_{R_1,R_2}(x_0) := \{x \in \mathbb{R}^N : R_1 < |x - x_0| < R_2\}.$$

In (1.1), for every $1 \leq i \leq n$, we suppose that $\alpha_i \in \mathcal{C}^\nu[0,\infty)$, for some $\nu \in (0,1]$, satisfy $\alpha_i \geq 0$ in Ω and $\alpha := (\alpha_1, \cdots, \alpha_n) \neq 0$. As far as concerns the nonlinear terms of (1.1), the following conditions are imposed:

(A1) For every $1 \leq i \leq n$, $f_i \in \mathcal{C}^{1+\nu}[0,\infty)$, $f_i(0) = 0$ and $f_i'(u) > 0$ for all $u > 0$.
(A2) There exists $F \in \mathcal{C}^{1+\nu}[0,\infty)$ such that $F(0) = 0$, $F(u) > 0$, $F'(u) > 0$ for all $u > 0$, and

$$\min_{1\leq i\leq n} f_i(u) \geq F(u) \quad \text{for all } u \geq 0, \qquad \lim_{u\uparrow\infty} F(u) = \infty.$$

In the singular case when $M = \infty$, a function

$$u = (u_1, \cdots, u_n) \in [\mathcal{C}^{2+\nu}(\Omega)]^n$$

is a solution of (1.1) if it satisfies the system and

$$\lim_{d(x)\downarrow 0} u_i(x) = \infty \quad \text{for all } 1 \leq i \leq n.$$

In such case, u is said to be a *large, or explosive, solution* of (1.1).

Under these general hypotheses, the problem (1.1) has a unique positive solution for every $0 < M < \infty$, denoted by $\theta_{[\lambda,\Omega,M]}$ with $\lambda := (\lambda_1, \ldots, \lambda_n)$. Moreover, by the maximum principle, $M \mapsto \theta_{[\lambda,\Omega,M]}$ is point-wise increasing. Thus, the limiting function

$$\theta_{[\lambda,\Omega,\infty]} := \lim_{M\uparrow\infty} \theta_{[\lambda,\Omega,M]} \tag{1.3}$$

is well defined, though it might equal ∞ somewhere in Ω, unless the following generalized Keller–Osserman condition, [12,28], holds

(KO) Assumptions (A1) and (A2) hold and, for every $\beta > 0$, $\lim_{u\uparrow\infty} I(u) = 0$, where

$$I(u) := \int_1^\infty \frac{d\theta}{\sqrt{\int_1^\theta [\beta F(ut) - 1] t dt}}.$$

See [19, Chapter 3] for a detailed discussion about condition (KO). In particular, under the assumption (A1), the condition (KO) holds if there are $p > 0$ and $C > 0$ such that

$$f_i(u) \geq F(u) := Cu^p \qquad \text{for all } u \geq 0 \quad \text{and} \quad 1 \leq i \leq n. \tag{1.4}$$

The main result of this paper is the following

Theorem 1.1. *Suppose* (KO)*,* $\Omega \in \{B_R(x_0), A_{R_1,R_2}(x_0)\}$*,* $\lambda_i \geq 0$*,* $a_{ij} > 0$ *for all* $1 \leq i \leq j \leq n$*,* $j \neq i$*,* α_i *are positive nondecreasing functions,*

$$0 < \alpha_i(t) \leq \alpha_i(s) \quad \textit{for all} \quad 0 < t \leq s, \quad 1 \leq i \leq n, \tag{1.5}$$