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Symmetry properties for nonnegative solutions of non-uniformly elliptic equations in the hyperbolic space ☆

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A B S T R A C T

We are interested in monotonicity and symmetry properties for nonnegative solutions of elliptic equations defined in geodesic balls of the hyperbolic space $\mathbb{H}^n$, which is the simplest example of manifold with negative curvature. More precisely, let B be a geodesic ball in $\mathbb{H}^n$ and let $u \in W^{1,p}(B) \cap L^\infty(B)$ be a sufficiently regular solution of $\Delta_p u + f(u) = 0$ in B with boundary condition $u = 0$, where Δ_p is the p-Laplace–Beltrami operator with $p > 2$. Then we prove local or global symmetry results for nonnegative solutions according to the assumptions about the zeros of the nonlinearity $f(s)$, which is merely continuous.

1. Introduction

The main purpose of this paper is to apply a variant of the method of moving planes and local inversion method to prove monotonicity and local or global symmetry results for nonnegative solutions for a class of quasilinear elliptic equations defined in geodesic balls of the hyperbolic space $\mathbb{H}^n$, and when the nonliearity is merely continuous.

1.1. Motivation and previous results

In order to motivate our results we begin by giving a brief survey on this subject. The study of symmetry properties for solutions of differential equations was started by J. Serrin [30] in 1971 by using the method of moving planes (MMP), also known as Alexandrov reflection method, created by the Soviet mathematician A.D. Alexandrov in the early 1950s to study manifolds with constant mean curvature (see [1,2]). Later, in

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celebrated papers [22,23], by using the MMP, B. Gidas, M. Ni and L. Nirenberg proved that any positive solution $u \in C^2(\overline{\Omega})$ of the problem $-\Delta u = f(u)$ in Ω with $u = 0$ on $\partial\Omega$ is radially symmetric when $f(s)$ is C^1 and $\Omega = B \subset \mathbb{R}^n$ is a ball or $\Omega = \mathbb{R}^n$ (assuming that $u(x) = O(|x|^{2-n})$ at infinity). After that, by using method of moving planes in combination with maximum principle for narrow domains, H. Berestycki and L. Nirenberg [5] improved the results in [22,23]. Explicitly, they proved monotonicity and symmetry in the x_1 direction for positive solutions $u \in W^{2,n}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ of nonlinear elliptic equations in a general bounded domain in $\Omega \subset \mathbb{R}^n$ which is convex in the x_1 direction by assuming that $f(s)$ is only Lipschitz continuous.

The method of moving planes and its variants have been applied to extend and improve the results cited above in some directions. First, it has been used to obtain symmetry properties for nonnegative solutions of semilinear elliptic equations involving continuous nonlinearities, not necessarily locally Lipschitz defined in bounded or unbounded domains of $\mathbb{R}^n$ (see for example [8–10,18–20] and references therein).

In another direction MMP has been used to study symmetry properties for positive solutions of p-Laplacian equations of the form

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$. It should be mentioned that the ideas used to study the Laplacian case cannot be applied directly to analyze symmetry for solutions of (1.1) because the p-Laplacian is a singular or degenerate elliptic operator if $1 < p < 2$ or $p > 2$, respectively, on the critical set

$$Z := \{x \in \Omega : \ \nabla u(x) = 0\},$$

and in general $C^{1,\alpha}$ is the optimal regularity result that we have for solutions of (1.1) as E. DiBenedetto, P. Tolksdorf have established in [17,31]. In this direction M. Badiale and E. Nabana [4] proved that positive solutions u of (1.1) are radially symmetric by assuming that $f(s)$ is a C^1-function, $f'(0) > 0$, $\Omega = B \subset \mathbb{R}^n$ is a ball or $\Omega = \mathbb{R}^n$ (assuming that $u(x) \to 0$ at infinity) and under the additional assumption that the origin is the unique critical point of u, that is, $Z = \{0\}$. They applied the MMP directly because under these assumptions u belongs to class C^2 in $B \setminus \{0\}$ and satisfies a second order uniformly elliptic equation. Later, this result was improved for the case $1 < p < 2$ in [11–13,15,16] without assuming the condition $Z = \{0\}$. For that it was crucial in their arguments a comparison principle due to L. Damascelli in [11]. For $p > 2$, by using a weighted Sobolev space with the weight $\rho = |Du|^{p-2}$ and the method of moving planes in combination with comparison principles, L. Damascelli and B. Sciunzi [14,15] proved monotonicity and symmetry properties for positive solutions $u \in C^1(\overline{\Omega})$ of problem (1.1), when $\Omega \subset \mathbb{R}^n$ is a symmetric bounded smooth domain and $f(s)$ is a positive and locally Lipschitz continuous function. In [29] B. Sciunzi extended the results in [15] to the case when $f(s)$ is allowed to change sign, but is nondecreasing near its zeros. We emphasize that case $p > 2$ is more involved. For instance, M. Grossi et al. [24] and F. Brock [7] gave examples of nonsymmetric solutions of the problem (1.1) for C^2 nonlinearities which change sign. Using new rearrangement techniques, called continuous Steiner symmetrization, F. Brock [6,7] proved symmetry results for nonnegative solutions of the problem (1.1) in the case $1 < p \leq 2$. He introduced a notion of *local symmetry* and proved that in case $p > 2$ any solution for (1.1) is locally radially symmetric if $f(s)$ is nondecreasing. The variant of the method of moving planes together with local inversion methods was used by J. Dolbeault, P. Felmer and R. Monneau [21] to obtain local symmetry results if $f(s)$ is merely continuous, has a finite number of zeros and is nonincreasing or nonnegative near its zeros. Moreover, they obtained global symmetry results if $f(s)$ is just continuous and positive.

Symmetry results for elliptic problems defined on Riemannian manifolds with constant sectional curvature were also considered recently by S. Kumaresan and J. Prajapat. More precisely, in [25] and [26] they have obtained analogous results to those of [22] and [30] for symmetric domains of the hyperbolic space $\mathbb{H}^n$ or the Sphere $\mathbb{S}^n$, respectively. To prove those results they have introduced an intrinsic geometric interpretation