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Higher integrability for solutions to parabolic problems with irregular obstacles and nonstandard growth

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A B S T R A C T

The aim of this paper is to derive the self-improving property of integrability for the spatial gradient of solutions to degenerate parabolic obstacle problem with irregular obstacles and $p(x,t)$-nonstandard growth. More precisely, we prove that the spatial gradient of the solution is integrable to a larger power than the natural one determined by the structural assumptions on the involved differential operator.

1. Introduction

In this paper we establish higher integrability properties of solutions to degenerate parabolic obstacle problems with $p(x,t)$-nonstandard growth, i.e. solutions to parabolic variational inequality satisfying an obstacle constraint. In general, the idea of the self-improving property of integrability is the following: In principle the proof is based on certain reverse Hölder inequalities and an application of the Gehring's Lemma. To conclude a reverse Hölder inequality, we need a Caccioppoli estimate. Note that a Caccioppoli estimate has the structure of a reverse Poincaré inequality. Such a Caccioppoli estimate follows by considering the weak formulation of the elliptic or parabolic equation resp. system, then applying the structure condition on the vector-field and an application of Sobolev–Poincaré inequality. This yields the desired Caccioppoli estimate and therefore, the reverse Hölder inequality. In the nonstandard case, it is necessary to use additionally a localization argument, which allows to homogenize the estimates, to derive a reverse Hölder type inequality. This estimate is comparable to the one from the standard case and the key to the higher integrability.

Historical background. In the elliptic case with standard p-growth, it is by now a classical fact that weak solutions are locally higher integrable in the sense of Meyer's higher integrability result for the spatial

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gradient. This was first proved for the Jacobian of quasiconformal mapping by Gehring [28] and later on, for solutions to elliptic systems by Elcrat and Meyers [21], see also the monograph [29]. The result of higher integrability is already stated for parabolic systems with standard p-growth by Kinnunen and Lewis [35]. In the stationary nonstandard case, there are results of higher integrability by Zhikov in [43], while in the nonstandard $p(z)$-growth case there is the higher integrability result for the homogeneous parabolic $p(z)$-Laplacian, i.e.

$$\partial_t u - \operatorname{div}(|Du|^{p(z)-2} Du) \ = 0 \text{ in } \Omega_T$$

by Antontsev and Zhikov in [6]. Moreover, there is the $p(z)$-analogue to [35] on the one hand by Bögelein and Duzaar [11] and on the other hand by Zhikov and Pastukhova in [44]. Zhikov and Pastukhova established independently and slightly earlier a higher integrability result, which is very similar to the one of Bögelein and Duzaar in [11]. Bögelein and Duzaar have shown a Meyer's type higher integrability result for the spatial gradient of weak solutions to parabolic systems of the form

$$\partial_t u - \operatorname{div} a(z, Du) \ = \ \operatorname{div}(|F|^{p(z)-2} F) \text{ in } \Omega_T. \tag{1.1}$$

Their result ensures that weak solutions of the preceding equation belong to a slightly higher Sobolev space than the natural space uniquely by the growth of the vector-field $a(z, \cdot)$ and therefore, obey a certain self-improving property of integrability. This result we may also extend to solutions to parabolic equations of the from:

$$\partial_t u - \operatorname{div} a(z, Du) = f - \operatorname{div}(|F|^{p(z)-2} F) \text{ in } \Omega_T. \tag{1.2}$$

Finally, we would like to mention that the higher integrability of solution to obstacle problems with p-growth, is a result by Bögelein and Scheven [12].

Motivation of parabolic problems with variable exponents and obstacles. Obstacle problems are interesting objects in the theory of partial differential equations and the calculus of variations. In general, the theory of obstacle problems is motivated by numerous applications, e.g. in mechanics or in control theory. We refer to [10,34] for an overview of the classical theory and applications. Moreover, obstacle problems have been exploited in nonlinear potential theory for approximating supersolutions by solutions to obstacle problems, see [31,33,36]. Up to now, the theory for elliptic problems is well understood, as well the theory for elliptic obstacle problems and also the nonstandard case. Therefore, parabolic problems arouse interest more and more in mathematics during the last years. Also parabolic problems are motivated by physical aspects. In particular, evolutionary equations and systems can be used to model physical processes, e.g. heat conduction or diffusion processes. There are many open problems, e.g. the Navier–Stokes equation, the basic equation of fluid mechanics. Furthermore, some properties of solutions of the system of a modified Navier–Stokes equation, describing electro-rheological fluids are studied in [3]. Such fluids are recently of high technological interest, because of their ability to change the mechanical properties under the influence of exterior electro-magnetic field, see [27,38]. For example, many electro-rheological fluids are suspensions consisting of solid particles and a carrier oil. These suspensions change their material properties dramatically if they are exposed to an electric field, see [39]. Most of the known results concern the stationary models, see for example [1,2]. Other applications are the models for flows in porous media [5,32].

Turning towards obstacle problems, one observes that the stationary case with standard growth is well developed, also the nonstandard case. Furthermore, in the last four or five years, a gap in the parabolic theory of obstacle problems with standard p-growth was closed, see [8,12,13,16–18,24,41]. Moreover, in the last two or three years there were several regularity results for the nonstandard growth case developed, see [7,9,19,22,23,25,42].