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Products of truncated Hankel operators ☆

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We characterize the pairs of truncated Hankel operators on the model spaces K_u^2 $(= H^2 \ominus uH^2)$ whose products result in truncated Toeplitz operators when the inner function u has a certain symmetric property.

1. Introduction

In 2007, D. Sarason defined truncated Toeplitz operators (TTO) as the compression of Toeplitz operators to invariant subspaces for the backward shift on the Hardy space H^2. Toeplitz matrices can be interpreted as truncated Toeplitz operators on finite dimensional model spaces. Recently, C. Gu defined truncated Hankel operators (THO) as the compression of Hankel operators to invariant subspaces for the backward shift and proved a number of algebraic properties of them. Some of the properties in his paper reveal the relation between the THO's and TTO's. In this paper, we will consider when the product of two THO's becomes a TTO.

Let $L^2 \equiv L^2(\mathbb{T})$ be the set of all square-integrable functions on the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$ and $H^2 \equiv H^2(\mathbb{T})$ be the corresponding Hardy space, i.e., the closed linear span of the analytic polynomials in L^2. The space H^∞ is defined by $H^\infty := H^2(\mathbb{T}) \cap L^\infty(\mathbb{T})$. A function $\theta \in H^\infty$ is called *inner* if $|\theta(z)| = 1$ almost everywhere on the unit circle T.

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For $\phi \in L^\infty$, the Toeplitz operator T_ϕ on H^2 is defined by

$$T_\phi f = P(\phi f),$$

where P is the orthogonal projection of L^2 onto H^2. The Hankel operator $H_\phi : H^2 \longrightarrow H^2$ with symbol $\phi \in L^\infty$ is defined by

$$H_\phi f = J(I - P)(\phi f),$$

where J denotes the unitary map on L^2 defined by $(Jf)(z) = \overline{z} f(\overline{z})$.

For a nonconstant inner function u, define the model space K_u^2 by

$$K_u^2 := H^2 \ominus uH^2.$$

It is known that the dimension of K_u^2 is finite if and only if u is a finite Blaschke product and in that case, $\dim K_u^2$ equals the number of zeros of u counting multiplicity. The dimension of K_u^2 is also called the *degree* of the inner function u and is denoted by *deg u*. If u is not a finite Blaschke product, we say that the degree of u is infinite. The following set equality is easily verified and is used occasionally in this paper:

$$K_u^2 = u\overline{zK_u^2}. \tag{1}$$

For a function $\phi \in L^2(\mathbb{T})$, the *truncated Toeplitz operator* A_ϕ on K_u^2 is defined by

$$A_\phi f = P_u(\phi f), \text{ for } f \in K_u^2,$$

where P_u denotes the orthogonal projection of L^2 onto K_u^2. For a function $\phi \in L^2(\mathbb{T})$, a *truncated Hankel operator* B_ϕ on K_u^2 is defined by

$$B_\phi f = P_u J(I - P)\phi f, \text{ for each } f \in K_u^2.$$

It is easy to see that B_ϕ does not depend on the analytic part of the symbol function ϕ. So, we often assume $\phi \in \overline{zH^2}$ when ϕ is the symbol function of a truncated Hankel operator. If $\phi \in L^2(\mathbb{T})$ is not an essentially bounded function, then A_ϕ or B_ϕ can be an unbounded operator. Since we are mainly concerned with bounded operators, we denote the set of all bounded truncated Toeplitz operators by $\mathfrak{T}(K_u^2)$ and the set of all bounded truncated Hankel operators by $\mathfrak{H}(K_u^2)$.

If the inner function u is z^n, then $\{1, z, z^2, \cdots, z^{n-1}\}$ forms an orthonormal basis for K_u^2. With respect to this basis, A_ϕ and B_ϕ can be represented as a Toeplitz matrix and a Hankel matrix, respectively:

$$A_\phi = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \ddots & \vdots \\ a_2 & a_1 & a_0 & \ddots & a_{-2} \\ \vdots & \ddots & \ddots & \ddots & a_{-1} \\ a_{n-1} & \cdots & a_2 & a_1 & a_0 \end{pmatrix}$$

and