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Note

Hypercyclic operators are subspace hypercyclic

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ABSTRACT

In this short note, we prove that for a dense set $\mathcal{A} \subset \mathcal{X}$ ($\mathcal{X}$ is a Banach space) there is a non-trivial closed subspace $\mathcal{M} \subset \mathcal{X}$ such that $\mathcal{A} \cap \mathcal{M}$ is dense in $\mathcal{M}$. We use this result to answer a question posed in Madore and Martínez-Avendaño (2011) [9]. In particular, we show that every hypercyclic operator is subspace-hypercyclic.

1. Introduction

A bounded linear operator T on a separable Banach space $\mathcal{X}$ is hypercyclic if there is a vector $x \in \mathcal{X}$ such that $Orb(T, x) = \{T^n x : n \geq 0\}$ is dense in $\mathcal{X}$, such a vector x is called hypercyclic for T. The first example of a hypercyclic operator on a Banach space was constructed by Rolewicz in 1969 [12]. He showed that if B is the backward shift on $\ell^p(\mathbb{N})$ then λB is hypercyclic if and only if $|\lambda| > 1$.

The studying of the scaled orbit and disk orbit is motivated by the Rolewicz example [12]. In 1974, Hilden and Wallen [5] defined the concept of supercyclicity. An operator T is called supercyclic if there is a vector x such that the scaled orbit $\mathbb{C}Orb(T, x)$ is dense in $\mathcal{X}$. Similarly, an operator T is called diskcyclic if there is a vector $x \in \mathcal{X}$ such that the disk orbit $\mathbb{D}Orb(T, x)$ is dense in $\mathcal{X}$, and such a vector x is called diskcyclic for T. For more information about diskcyclic operators, the reader may refer to [4] and [3].

In 2011, Madore and Martínez-Avendaño [9] considered the density of the orbit in a non-trivial subspace instead of the whole space, this phenomenon is called as subspace-hypercyclicity. An operator is called subspace-hypercyclic (or $\mathcal{M}$-hypercyclic, for short) for a subspace $\mathcal{M}$ of $\mathcal{X}$ if there exists a vector such that the intersection of its orbit and $\mathcal{M}$ is dense in $\mathcal{M}$. This concept has been studied in several papers, for example [11] and [6].

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In the same paper, Madore and Martínez-Avendaño showed that subspace-hypercyclicity does not imply to hypercyclicity (see [9, Example 2.2]). However they asked a question whether hypercyclicity implies to subspace-hypercyclicity for some subspaces. In the same year, Le [7] answered this question but under certain conditions. In 2015, Martínez-Avendaño and Zatarain-Vera [10] proved that hypercyclic coanalytic Toeplitz operators are subspace-hypercyclic under certain conditions. However, the problem of whether every hypercyclic operator is subspace-hypercyclic is still an open problem. Therefore, we answer this problem affirmatively in this paper.

In Section 2, we prove that if $\mathcal{A}$ is a dense subset of a Banach space $\mathcal{X}$, then there is a non-trivial closed subspace $\mathcal{M}$ of $\mathcal{X}$ such that $\mathcal{A} \cap \mathcal{M}$ is dense in $\mathcal{M}$. As a consequence, we show that if T is hypercyclic, then T is $\mathcal{M}$-hypercyclic which answers the question (iii) that was posed by Madore and Martínez-Avendaño in [9]. As immediate consequences, we get some further properties of subspace-hypercyclic operators.

2. Main result

The following theorem is our main result.

Theorem 2.1. *If $\mathcal{A}$ is a dense subset of a Banach space $\mathcal{X}$, then there is a non-trivial closed subspace $\mathcal{M}$ such that $\mathcal{A} \cap \mathcal{M}$ is dense in $\mathcal{M}$.*

To prove Theorem 2.1 above, we need the following two lemmas.

Lemma 2.2. *Let e be a fixed element in $\mathcal{X}$ such that $\|e\| > 1$. Then, for every $\epsilon > 0$ there is a non-empty open subset U of $\mathcal{X} \backslash \{0\}$, such that for all $x \in U$*

$$\|x\| < \epsilon \text{ and } dist\,(e, lin\{x\}) > 1.$$

Proof. Let us fix $f \in \mathcal{X}^*$ with $\|f\| = 1$ and $f(e) = \|e\|$, and let

$$U = \left\{ x \in \mathcal{X} \backslash \{0\} : \|x\| < \epsilon \text{ and } \frac{|f(x)|}{\|x\|} < \frac{\|e\| - 1}{1 + \|e\|} \right\}.$$

It is clear that U is an open subset of $\mathcal{X} \backslash \{0\}$ and norms of its elements are smaller than ϵ. Let us fix $x \in U$ and $t \in \mathbb{C}$, then we can consider the following two cases.

Case 1. If $|t| > \frac{1+\|e\|}{\|x\|}$. Then

$$\|e - tx\| \geq ||t|\,\|x\| - \|e\|| > 1.$$

Case 2. If $|t| \leq \frac{1+\|e\|}{\|x\|}$. Then, since $|f(e - tx)| \leq \|f\|\,\|e - tx\| \leq \|e - tx\|$, we have

$$\begin{aligned} \|e - tx\| \geq |f(e) - tf(x)| &\geq \|e\| - |tf(x)| \\ &\geq \|e\| - \frac{\|e\| + 1}{\|x\|}|f(x)| \\ &> \|e\| - (\|e\| + 1)\frac{\|e\| - 1}{\|e\| + 1} \\ &> 1. \end{aligned}$$

Thus, we get $dist\,(e, lin\{x\}) > 1$. □