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New local and parallel finite element algorithm based on the partition of unity

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ABSTRACT

In this study, based on a combination of the two-grid method and the partition of unity-based domain decomposition method, we propose a new local and parallel finite element algorithm for the elliptic boundary value problem. The proposed method has three key features: (1) it inherits the flexibility and controllability of domain decomposition based on the partition of unity; (2) global fine grid correction is replaced by solving a series of locally defined approximate residual problems with homogeneous Dirichlet boundary conditions on some finer grids; (3) a global continuous finite element solution is constructed by solving a coarse grid correction problem and by assembling all the local solutions together using the partition of unity subordinate. Under appropriate assumptions, the optimal error estimates in L^2 and the energy norms are proved by new analytical results. In addition, several numerical simulations are presented to demonstrate the high efficiency and flexibility of the new algorithm.

1. Introduction

Recently, local and parallel finite element computation methods have attracted much attention for solving partial differential equations. A local and parallel approach was first proposed by Xu and Zhou based on understanding the local and global properties of a finite element solution in their pioneering study [24]. This approach is based on local finite element discretization and it requires less communication between blocks than the standard Galerkin methods. Bank and Holst [4,5] developed a similar approach where

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the implementation details focused on adaptivity. Many other studies have addressed this issue for other problems. Interested readers are referred to recent studies by [10–12,20,21], and others.

The partition of unity method [2,18] provides a flexible and controllable method for performing domain decomposition. By combining the partition of unity method with the parallel adaptive algorithm, Holst [14,15] constructed the parallel partition of unity method. Huang and Xu [16] proposed a finite element discretization for elliptic boundary value problems by using a partition of unity method. Bacuta et al. [3] proposed a partition of unity refinement method for improving the local approximations of elliptic boundary value problems in the regions of interest. Wang et al. [23] also proposed the so-called "two-grid partition of unity method" by combining the two-grid method with the partition of unity method. The localization technique employed by [17] and [22] is also associated with the partition of unity method.

In the present study, based on the partition of unity method, we propose a local and parallel finite element algorithm for the following elliptic boundary value problem defined on a smooth domain $\Omega \subset R^d$, $d = 2, 3$:

$$\begin{aligned} -\Delta u + b \cdot \nabla u &= f \quad \texttt{in } \Omega, \\ u &= 0 \quad \texttt{on } \partial\Omega. \end{aligned} \tag{1.1}$$

First, we decompose the entire computational domain Ω into a series of disjoint small regions $\{D^i\}$. The global lower frequency approximation is then computed by the standard finite element method on a coarse grid, such as τ_H. We then post-process this standard finite element approximation by solving a series of locally defined approximate residual problems with homogeneous Dirichlet boundary conditions on some finer grids of $\{D^i\}$, or their extensions $\{D^{i,k}\}$, from which we obtain an approximation of the higher frequency component. Finally, a global continuous finite element solution is obtained by combining the independently produced solutions using a partition of unity $\{\phi_i\}_1^N$, which is subordinate to the overlapping subdomains $\{D^i\}$.

This parallel algorithm has several important features. First, the partition of unity introduces a framework for localizing the post-processing procedure, which requires far less communication. In addition, due to the partition of unity and its associated local computational subdomains, we can obtain a global continuous finite element solution. Moreover, multi-layer oversampling is introduced to enlarge the original subdomain D^i, thereby eliminating some of the detrimental effects on the global accuracy caused by the artificial homogeneous Dirichlet boundary conditions. Coarse grid correction is also used to construct the final solution to improve the global finite element approximation and to obtain the optimal convergence order.

The differences among the methods of Holst [15], Wang et al. [23] and our proposed method are summarized as follows.

• The error estimates are improved by the proposed method. In the recent local estimates by Xu and Zhou [24], the global error estimates are derived asymptotically by $O(\sqrt{P}(h + H^2))$ in H^1-norm, where P is the number of local spaces, which is denoted by N in this study. To improve this result, the partition of unity method analysis framework is replaced by two important lemmas: Lemmas 3.1 and 3.2. Based on these two lemmas, we obtain an improved error estimate of order $O(h + |\ln N|^{3-d} H^2)$ in the H^1-norm.

• To implement the algorithm, we introduce a framework for domain decomposition based on the partition of unity, which is derived directly from a regular triangulation and its finite element basis functions (see Section 3.3). Thus, the parallel algorithm can operate in a parallel environment at the element level.

The remainder of this paper is organized as follows. In Section 2, we introduce some preliminary details. The local and parallel finite element algorithm based on the partition of unity is proposed and analyzed in Section 3. In particular, a practical framework for the partition of unity is presented in this section. In Section 4, we describe the implementation details and some numerical simulations are presented to illustrate the efficiency of our method. We give our conclusions in the last section.