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Existence of invariant densities for semiflows with jumps ☆

Weronika Biedrzycka *, Marta Tyran-Kamińska

Institute of Mathematics, University of Silesia, Bankowa 14, 40-007 Katowice, Poland

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A B S T R A C T

The problem of existence and uniqueness of absolutely continuous invariant measures for a class of piecewise deterministic Markov processes is investigated using the theory of substochastic semigroups obtained through the Kato–Voigt perturbation theorem on the L^1-space. We provide a new criterion for the existence of a strictly positive and unique invariant density for such processes. The long time qualitative behavior of the corresponding semigroups is also considered. To illustrate our general results we give a detailed study of a two dimensional model of gene expression with bursting.

1. Introduction

We study a class of piecewise-deterministic Markov processes (PDMPs) which we call semiflows with jumps. As defined in [10,11] a PDMP without active boundaries is determined by three local characteristics $(\pi, \varphi, \mathcal{P})$, where π is a semiflow describing the deterministic parts of the process, $\varphi(x)$ is the intensity of a jump from x, and $\mathcal{P}(x, \cdot)$ is the distribution of the state reached by that jump. The problem of existence of invariant measures for Markov processes is of fundamental importance in many applications of stochastic processes [11,18,24].

We consider semiflows that arise as solutions of ordinary differential equations

$$x'(t) = g(x(t)), \tag{1.1}$$

where $g: \mathbb{R}^d \to \mathbb{R}^d$ is a (locally) Lipschitz continuous mapping. We assume that E is a Borel subset of $\mathbb{R}^d$ such that for each $x_0 \in E$ the solution $x(t)$ of (1.1) with initial condition $x(0) = x_0$ exists and that $x(t) \in E$ for all $t \geq 0$. We denote this solution by $\pi_t x_0$. Then the mapping $(t, x_0) \mapsto \pi_t x_0$ is Borel measurable and satisfies $\pi_0 x = x$, $\pi_{t+s} x = \pi_t(\pi_s x)$ for $x \in E$, $s, t \in \mathbb{R}_+$. As concerns jumps we consider a family of

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* Corresponding author.
E-mail addresses: wsiwek@us.edu.pl (W. Biedrzycka), mtyran@us.edu.pl (M. Tyran-Kamińska).

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measurable transformations $T_\theta\colon E \to E$, $\theta \in \Theta$, where Θ is a metric space which carries a Borel measure ν, and a family of measurable functions $p_\theta\colon E \to [0, \infty)$, $\theta \in \Theta$, satisfying

$$\int_\Theta p_\theta(x)\nu(d\theta) = 1, \quad x \in E,$$

so that the stochastic kernel $\mathcal{P}$ is of the form

$$\mathcal{P}(x, B) = \int_\Theta 1_B(T_\theta(x))p_\theta(x)\nu(d\theta), \quad x \in E, \tag{1.2}$$

for $B \in \mathcal{B}(E)$, where $\mathcal{B}(E)$ is the Borel σ-algebra of subsets of E. This roughly means that if the value of the process is x then we jump to the point $T_\theta(x)$ with probability $p_\theta(x)$.

The following standing assumptions will be made. The intensity function φ is continuous and

$$\lim_{t\to\infty} \int_0^t \varphi(\pi_s x)ds = +\infty \quad \text{for all } x \in E. \tag{1.3}$$

The mappings $(\theta, x) \mapsto T_\theta(x)$ and $(\theta, x) \mapsto p_\theta(x)$ are measurable so that the stochastic kernel in (1.2) is well defined. We assume also that each mapping $\pi_t\colon E \to E$ as well as each $T_\theta\colon E \to E$ is nonsingular with respect to a reference measure m on E. Recall that a measurable transformation $T\colon E \to E$ is called *nonsingular* with respect to m if the measure $m \circ T^{-1}$ is absolutely continuous with respect to m, i.e., $m(T^{-1}(B)) = 0$ whenever $m(B) = 0$.

Let us briefly describe the construction of the PDMP $\{X(t)\}_{t\geq 0}$ with characteristics $(\pi, \varphi, \mathcal{P})$ (see e.g. [10, 11] for details). Define the function

$$F_x(t) = 1 - \exp\{-\int_0^t \varphi(\pi_s x)ds\}, \quad t \geq 0,\ x \in E, \tag{1.4}$$

and note that the assumptions imposed on φ imply that F_x is a distribution function of a positive and finite random variable for every $x \in E$. Let $t_0 = 0$ and let $X(0) = X_0$ be an E-valued random variable. For each $n \geq 1$ we can choose the nth *jump time* t_n as a positive random variable satisfying

$$\Pr(t_n - t_{n-1} \leq t | X_{n-1} = x) = F_x(t), \quad t \geq 0,$$

and we define

$$X(t) = \begin{cases} \pi_{t-t_{n-1}}(X_{n-1}) & \text{for } t_{n-1} \leq t < t_n, \\ X_n & \text{for } t = t_n, \end{cases}$$

where the nth *post-jump position* X_n is an E-valued random variable such that

$$\Pr(X_n \in B | X(t_n-) = x) = \mathcal{P}(x, B),$$

and $X(t_n-) = \lim_{t\uparrow t_n} X(t) = \pi_{t_n-t_{n-1}}(X_{n-1})$. In this way, the trajectory of the process is defined for all $t < t_\infty := \lim_{n\to\infty} t_n$ and t_∞ is called the explosion time. To define the process for all times, we set $X(t) = \Delta$ for $t \geq t_\infty$, where $\Delta \notin E$ is some extra state representing a cemetery point for the process. The PDMP $\{X(t)\}_{t\geq 0}$ is called the *minimal* PDMP corresponding to $(\pi, \varphi, \mathcal{P})$. It is said to be *non-explosive* if