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# Overdetermined partial resolvent kernels for finite networks

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A B S T R A C T

In [2], a study of the existence and uniqueness of solution of partial overdetermined boundary value problems for finite networks was performed. These problems involve Schrödinger operators and the novel feature is that no data are prescribed on part of the boundary, whereas both the values of the function and of its normal derivative are given on another part of the boundary. In the present work, we study the resolvent kernels associated with overdetermined partial boundary value problems on finite network and we express them in terms of the well-known Green operator and the Dirichlet-to-Robin map. Moreover, we analyze their main properties and we compute them in the case of a generalized cylinder. The obtained expression involve polynomials that can be seen as a generalization of Chebyshev polynomials, and indeed when the conductances along axes are constant the expressions for the overdetermined partial resolvent kernels are given in terms of second kind Chebyshev polynomials.



## 1. Preliminaries

A *discrete inverse boundary value problem* consists in recovering the conductances of a network with boundary using only boundary measurements and global equilibrium conditions. In general, inverse problems are exponentially ill-posed, since they are highly sensitive to changes in the boundary data, see [11]. On the other hand, discrete inverse problems appear naturally when discretizing continuous inverse problems, see for instance [5]. Although the discrete inverse problem has been completely characterized in the case of the combinatorial Laplacian for planar critical networks, see [7,9], few works have been done for general networks, where the inverse problem remains open. In [12], an extension of the cited works has been developed for networks embedded in a cylindric surface.

This work is the third in a series on various aspects of the discrete inverse problem. It develops the study corresponding to resolvent kernels associated with overdetermined partial boundary value problems for Schrödinger operators on networks. The appropriate theoretical framework to address the discrete inverse

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problem is the study of overdetermined partial boundary value problems, while the fundamental tool is the Dirichlet-to-Robin map, which measures the difference of voltages between boundary vertices when electrical currents are applied to them. The theoretical foundations of this class of problems have been established in [2]. The results in this framework are of potential application among others, in electrical impedance tomography which is currently one of the non-invasive methods of clinical diagnosis with more development opportunities; see [6]. The data for an inverse problem on a network is the Dirichlet-to-Robin map, since it contains the boundary information, so we worried for their properties, which were analyzed in [3]. The matrix associated with the Dirichlet-to-Robin map is known as the response matrix of the network and it is the Schur complement of a certain submatrix of the Schrödinger matrix. The consideration of Schrödinger operators allowed us to consider a wide class of matrices, not necessarily singular nor weakly diagonally dominant, as response matrices. Therefore, our results represented a generalization of those obtained in [7,8].

In the study of classical boundary value problems one of the main tools, both for solving as for studying fundamental properties, are the resolvent kernels such as Green, Poisson or Robin kernels. So, once we have established the overdetermined partial boundary value problem, we raise the problem of defining what is a resolvent kernel and what are its main properties. In order to do this, we first obtain an equivalent condition for the existence and uniqueness of solution of these type of problems that can be read directly from a submatrix of the Schrödinger operator. Then, we give expressions of these kernels in terms of the classical Green kernel and the Dirichlet-to-Robin map. In the last section, we obtain the resolvent kernels for a generalized cylinder, which are defined as the product of a path with an arbitrary network. The expressions are given in terms of a generalization of Chebyshev polynomials for higher dimensions, that when the conductances are constant are precisely Chebyshev polynomials of second kind.

Let $\Gamma = (V, c)$ be a finite network, i.e., a finite connected graph without loops nor multiple edges, and with the vertex set $V$. Let $E$ be the set of edges of the network $\Gamma$. Each edge $(x, y)$ is assigned a *conductance* $c(x, y)$, where $c : V \times V \longrightarrow [0, +\infty)$. Moreover, $c(x, y) = c(y, x)$ and $c(x, y) = 0$ if $(x, y) \notin E$. Then, $x, y \in V$ are *adjacent*, $x \sim y$, iff $c(x, y) > 0$. We denote by $V(S)$ the *set of neighbors of* $S \subset V$; that is, the set of vertices of $V \setminus S$ adjacent to any vertex $x \in S$.

The set of functions on a subset $F \subseteq V$, denoted by $\mathcal{C}(F)$, and the set of nonnegative functions on $F$, $\mathcal{C}^+(F)$, are naturally identified with $\mathbb{R}^{|F|}$ and the nonnegative cone of $\mathbb{R}^{|F|}$, respectively. We denote by $\int\limits_F u(x)dx$ the value $\sum\limits_{x\in F} u(x)$. Moreover, if $F$ is a non-empty subset of $V$, its characteristic function is denoted by $\chi_F$. When $F = \{x\}$, its characteristic function will be denoted by $\varepsilon_x$. If $u \in \mathcal{C}(V)$, we define the *support* of $u$ as $\mathsf{supp}(u) = \{x \in V : u(x) \neq 0\}$. Clearly, $\mathcal{C}(F)$ can be identified with the subspace $\{u \in \mathcal{C}(V) : \mathsf{supp}(u) \subset F\}$.

If we consider a proper subset $F \subset V$, then its *boundary* $\delta(F)$ is given by the vertices of $V \setminus F$ that are adjacent to at least one vertex of $F$. The vertices of $\delta(F)$ are called *boundary vertices* and when a boundary vertex $x \in \delta(F)$ has a unique neighbor in $F$ we call the edge joining them a *boundary spike*. It is easy to prove that $\bar{F} = F \cup \delta(F)$ is connected when $F$ is. Any function $\omega \in \mathcal{C}^+(\bar{F})$ such that $\mathsf{supp}(\omega) = \bar{F}$ and $\int\limits_{\bar{F}} \omega^2(x)\,dx = 1$ is called *weight* on $\bar{F}$. The set of weights is denoted by $\Omega(\bar{F})$. We denote $\kappa_F \in \mathcal{C}(\delta(F))$ as the function $\kappa_F(x) = \sum\limits_{y\in F} c(x, y)$.

We define the *normal derivative of* $u \in \mathcal{C}(\bar{F})$ *on* $F$ as the function in $\mathcal{C}(\delta(F))$ given by

$$\left(\frac{\partial u}{\partial \mathsf{n}_F}\right)(x) = \int\limits_F c(x, y)\,\big(u(x) - u(y)\big)\,dy, \quad \text{for any } x \in \delta(F).$$

If $H, F$ are non-empty subsets of $V$, any function $K \in \mathcal{C}(H \times F)$ will be called a *kernel*. The *integral operator associated with* $K$ is the endomorphism $\mathscr{K} : \mathcal{C}(F) \longrightarrow \mathcal{C}(H)$ that assigns to each $f \in \mathcal{C}(F)$, the function