J. Math. Anal. Appl. 435 (2016) 133–161

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Higher order symmetries and integrating factors for ordinary differential equations

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ARTICLE INFO

Article history:
Received 7 July 2015
Available online 9 October 2015
Submitted by Goong Chen

Keywords:
Integrating factors
Symmetries
Ordinary differential equations

ABSTRACT

Symmetry analysis and conservation laws are widely used in analyzing and solving differential equations. Conservation laws are also called first integrals when dealing with ordinary differential equations (ODEs). We explore the complementary nature of symmetry analysis and conservation laws; specifically, the use of symmetries to find integrating factors and, conversely, the use of conservation laws to seek new symmetries. In particular, we show two new fundamental results. Firstly, we show that a higher-order symmetry of an ODE induces a Lie point symmetry of the corresponding integrating factor determining equations (IFDEs). Moreover, we obtain an explicit expression for the infinitesimal generator of this induced point symmetry, enabling one to find new integrating factors of existing ODEs and thus possibly to solve a wider range of ODEs. Secondly, we show that the converse also holds, i.e. a point symmetry of the IFDEs for a given ODE induces a higher-order symmetry of the ODE.

1. Introduction

The mathematical modeling of physical phenomena frequently leads to differential equations that have conserved quantities (*first integrals* for ordinary differential equations) such as mass, energy, momentum and charge. Numerically, first integrals are used in generating finite element methods as well as to check the accuracy of numerical schemes. Recently there have been advances in the development of systematic methods for finding first integrals, allowing one to obtain analytic solutions to many otherwise-intractable differential equations. Much of this recent work focuses on the development and use of symbolic computation methods to find symmetries and integrating factors for ODEs (see [7–9] and references therein).

Given an nth order ODE

$$y^{(n)} - f(x, y, y', \dots, y^{(n-1)}) = 0, \tag{1}$$

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http://dx.doi.org/10.1016/j.jmaa.2015.09.064

one finds its first integrals systematically by the related task of seeking integrating factors, functions which satisfy an over-determined linear PDE system called the *integrating factor determining equations* (IFDEs). For an integrating factor of (1), it is straightforward to calculate a corresponding first integral and reduce the order of the ODE. Due to the nature of the set of solutions of the IFDEs, it is seldom possible to obtain the general solution of the IFDEs. Since any integrating factor allows one to reduce (1) to an $(n-1)$th order ODE, the aim is to find particular solutions of the IFDEs. Given n functionally-independent integrating factors, in principle one can obtain the general solution of ODE (1).

One way to obtain particular solutions of an IFDE system is to consider a particular ansatz for integrating factors, such as

$$\Lambda(x, \underline{Y}) = \tilde{\Lambda}(x, Y, Y_1, \dots, Y_k) \tag{2}$$

where $k < n$. Another approach is to employ symmetry methods to find integrating factors either by seeking invariant solutions of the IFDEs [2], or by exploiting the fact that symmetries of the IFDE system map the set of first integrals to itself [6]. Thus from a particular first integral one can use a symmetry of the IFDEs to generate an infinite sequence of first integrals, some of which may be new [2,11]. The use of symmetries of the IFDE system thus presents a systematic way to search for integrating factors of an ODE (1).

In [2,11] it was shown that every contact symmetry of (1) gives rise to a transformation mapping each integrating factor into another, possibly distinct, integrating factor. Since every symmetry of (1) must induce a symmetry of the corresponding IFDE system, one sees that given a contact symmetry of (1) with infinitesimal generator

$$\mathbb{X} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta^i \frac{\partial}{\partial y_i}, \tag{3}$$

one can obtain an infinitesimal generator for the induced IFDE symmetry $(x, \underline{Y}, \Lambda)$, given by [2]

$$\hat{\mathbb{X}} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial Y} + \eta^i \frac{\partial}{\partial Y_i} + \left(\frac{\partial \eta^{n-1}}{\partial Y_{n-1}} \right) \Lambda \frac{\partial}{\partial \Lambda}. \tag{4}$$

Using this one seeks solutions of the IFDE system invariant under (4) using the method of differential invariants [2]. There are many examples [2,4,5] for which this technique has proven effective in obtaining integrating factors and corresponding first integrals. In addition, there has recently been much work in employing a similar idea to find multipliers and conservation laws, the analogues of integrating factors and first integrals, respectively, for partial differential equations [3,6,12,13].

In this paper we extend the above results to higher-order symmetries of (1), potentially increasing the number of integrating factors which can be generated by both methods outlined above. We also demonstrate the completeness of the ODE symmetry method for finding symmetries of the corresponding IFDEs. In particular, we show that all point symmetries of the IFDE system which are of the form

$$\mathbb{X} = \xi(x, \underline{Y}) \frac{\partial}{\partial x} + \eta(x, \underline{Y}) \frac{\partial}{\partial Y} + \sum_{j=1}^{n-1} \eta^j(x, \underline{Y}) \frac{\partial}{\partial Y_j} + \Lambda h(x, \underline{Y}) \frac{\partial}{\partial \Lambda}$$

project onto symmetries of (1). Hence one does not need to construct and solve the symmetry determining equations of the IFDEs to ensure that all useful symmetries have been found.

In Section 2 we outline the construction of the integrating factor determining equations and introduce convenient notation for describing symmetries of the IFDEs. In Section 3 we restrict our attention to first-order ODEs, proving directly that all symmetries of the IFDEs of a given first order ODE project onto symmetries of the ODE. Next, in Section 4 we consider the general case for an arbitrary nth order ODE