J. Math. Anal. Appl. 435 (2016) 267–285

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Abstract Cauchy problem for weakly continuous operators

Toshitaka Matsumoto*, Naoki Tanaka

Department of Mathematics, Faculty of Science, Shizuoka University, Shizuoka 422-8529, Japan

ARTICLE INFO

Article history:
Received 12 June 2015
Available online 19 October 2015
Submitted by P.G. Lemarie-Rieusset

Keywords:
Abstract Cauchy problem
Weakly continuous operator
Growth condition
Comparison function
Maximal solution

ABSTRACT

We discuss the solvability of the abstract Cauchy problem $u'(t) = Au(t)$ for $t > 0$ and $u(0) = u_0 \in D$, where D is a weakly closed subset of a Banach space Y continuously embedded in the underlying Banach space X and A is a nonlinear operator from D into X satisfying a local weak continuity condition described by a family of functionals. We introduce a new type of weak compactness on the level sets of functionals and give a necessary and sufficient condition for the existence of a weakly continuously differentiable solution satisfying a growth condition. We apply the abstract result obtained to the Cauchy problem for a nonlinear Schrödinger equation and a mixed problem for a logarithmic wave equation.

1. Introduction

This paper is devoted to the solvability of the abstract Cauchy problem

$$u'(t) = Au(t) \text{ for } t > 0 \quad \text{and} \quad u(0) = u_0 \in D,$$

where D is a weakly closed subset of a Banach space Y continuously embedded in the underlying Banach space X and A is a nonlinear operator from D into X satisfying a weak continuity condition. We are interested in finding a necessary and sufficient condition for the existence of a weakly continuously differentiable solution satisfying a growth condition.

In the special case where Y is a Hilbert space and A is a weakly sequentially continuous mapping on Y to X such that $\langle v, Av\rangle_{V,X} \le \beta(\|v\|_Y^2)$ for $v \in V$, where $\beta \in C([0,\infty); [0,\infty))$ and V is another Banach space continuously embedded in Y such that $\langle v, u\rangle_{V,X} = (v, u)_Y$ for $v \in V$ and $u \in Y$, Kato and Lai [9] established the existence of a weakly continuously differentiable solution u satisfying the growth condition that $\|u(t)\|_Y^2 \le \rho(t)$, where ρ is the maximal solution of the ordinary differential equation $r'(t) = \beta(r(t))$ for $t \ge 0$, and applied their abstract result to the Euler equation. Their abstract result is based on the following

* Corresponding author.
E-mail addresses: matsumoto.toshitaka@shizuoka.ac.jp (T. Matsumoto), tanaka.naoki@shizuoka.ac.jp (N. Tanaka).

http://dx.doi.org/10.1016/j.jmaa.2015.10.027
0022-247X/

two key formulations, and it has significant applications to fluid dynamics ([15,16], see also [5]), and to the KdV equation (see [14]).

(i) A functional φ defined only by the original norm of Y is used to specify the growth of solutions.
(ii) The level sets of φ are weakly compact in Y.

Their result was extended by Günther and Prokert [6] in such a way that a family $\{\langle\cdot,\cdot\rangle_w; w \in W\}$ of bilinear forms was introduced by using an open set W in Y and the functional $\varphi(u) = (u,u)_u^{1/2}$ for $u \in W$ was used for the same purpose, where $\{(\cdot,\cdot)_w; w \in W\}$ is a family of inner products of Y compatible with the family of bilinear forms.

For wider applications, it is necessary to extend their results for the following reasons. For an application to a logarithmic wave equation, another type of functional was used by Han [7] to estimate the growth of solutions. To study the nonlinear Schrödinger equation

$$iu_t + \Delta u + |u|^{q-1}u = 0,$$

two functionals $\varphi_1(v) = \|v\|_{L^2}^2$ for $v \in L^2(\mathbb{R}^N)$ and $\varphi_2(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{1}{q+1}\|v\|_{L^{q+1}}^{q+1}$ for $v \in H^1(\mathbb{R}^N)$ were used by Ginibre and Velo [4] under the situation that the topology of $L^2(\mathbb{R}^N)$ is stronger than that of $L^{q+1}(\mathbb{R}^N)$ on any bounded subset of $H^1(\mathbb{R}^N)$. In this case, the set $\{v; \varphi_j(v) \le \alpha_j$ for $j = 1,2\}$ is not weakly closed in $H^1(\mathbb{R}^N)$, and so condition (ii) is not satisfied. It should be noted that the solution u of this equation satisfies the conservation law of the charge $\varphi_1(u(t)) = \varphi_1(u_0)$. By using the functional φ_1, a weak closedness is recovered in a sense that if $u \in H^1(\mathbb{R}^N)$ and $\{u_n\}$ is a sequence in $H^1(\mathbb{R}^N)$ converging weakly in $H^1(\mathbb{R}^N)$ to u as $n \to \infty$ and if $\limsup_{n\to\infty} \varphi_1(u_n) \le \varphi_1(u)$, then $\varphi_2(u) \le \liminf_{n\to\infty} \varphi_2(u_n)$.

The above consideration leads us to a new type of weak compactness condition ((φ1) and (φ2) in the next section) of level sets of a family of functionals combined with an abstract energy identity. To formulate an energy identity, we use a d-tuple $\varphi = (\varphi_j)_{j=1}^d$ of functionals from X into $[0,\infty]$ and a comparison function g (see condition (φ2-a)). The main result (Theorem 2.1) gives an extension of [9, Theorem A] and [6, Theorem 3.4].

As applications of the main result we discuss the Cauchy problem for a nonlinear Schrödinger equation and a mixed problem for a logarithmic wave equation in Sections 4 and 5. In the final section we derive Kato and Lai's existence result from Corollary 2.3.

2. Main theorem

Let $(X, \|\cdot\|_X)$ be a Banach space and let $(Y, \|\cdot\|_Y)$ be another Banach space continuously embedded in X. Then we discuss the solvability of the abstract Cauchy problem

$$u'(t) = Au(t) \text{ for } t > 0 \quad \text{and} \quad u(0) = u_0 \in D, \tag{CP}$$

where D is a weakly closed subset of Y and A is a nonlinear operator from D into X satisfying a weak continuity condition formulated as follows:

Let $\varphi = (\varphi_i)_{i=1}^d$ be a d-tuple of functionals from X into $[0,\infty]$ such that $D \subset D(\varphi) := \{v \in X; \varphi_j(v) < \infty$ for $1 \le j \le d\}$, and then we introduce the following weak continuity condition in a local sense:

(A) If $\alpha \in \mathbb{R}_+^d$, $u_n \in D_\alpha(\varphi)$ for $n \ge 1$, $u \in D$ and $u_n \to u$ weakly in Y as $n \to \infty$, then $Au_n \to Au$ weakly in X as $n \to \infty$.