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Strictly positive definite kernels on a product of spheres

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A B S T R A C T

For the real, continuous, isotropic and positive definite kernels on a product of spheres, one may consider not only its usual strict positive definiteness but also strict positive definiteness restrict to the points of the product that have distinct components. In this paper, we provide a characterization for strict positive definiteness in these two cases, settling all the cases but those in which one of the spheres is a circle.

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1. Introduction

Let S^m denote the m-dimensional unit sphere in $\mathbb{R}^{m+1}$ and S^∞ the unit sphere in the real space ℓ^2. Extrapolating a little bit the concepts found in [13], but still keeping the setting of the general theory developed in [3], we will say that a kernel $K : S^m \times S^M \to \mathbb{R}$ is *positive definite* if

$$\sum_{\mu,\nu=1}^{n} c_\mu c_\nu K((x_\mu, w_\mu), (x_\nu, w_\nu)) \geq 0,$$

for $n \geq 1$, distinct points $(x_1, w_1), (x_2, w_2), \ldots, (x_n, w_n)$ on $S^m \times S^M$, and real scalars $c_1, c_2, \ldots, c_n$. It is *strictly positive definite* if it is positive definite and the previous inequalities are strict whenever at least one of the c_μ is nonzero. It is *DC-strictly positive* if it is positive definite and the previous inequalities are strict whenever the x and the w components of the points are pairwise distinct and at least one of the c_μ is nonzero. Clearly, a strictly positive definite kernel is DC-strictly positive definite but not conversely. The symbol *DC* refers to “distinct components”.

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A kernel K acting on $S^m \times S^M$ is *isotropic* (*radial*) if

$$K((x,z),(y,w)) = K_r(x \cdot y, z \cdot w), \quad x,y \in S^m, \quad z,w \in S^M,$$

for some real function K_r on $[-1,1]^2$, where $\cdot$ stands for the inner product of both $\mathbb{R}^{m+1}$ and $\mathbb{R}^{M+1}$. In other words, the isotropy of K corresponds to the property

$$K((Ax,Bz),(Ay,Bw)) = K((x,z),(y,w)),$$

for $x,y \in S^m$, $z,w \in S^M$, $A \in \mathcal{O}^m$, and $B \in \mathcal{O}^M$, in which $\mathcal{O}^m$ denotes the orthogonal group in $\mathbb{R}^{m+1}$. The kernel K_r in the definition above will be referred to as the *isotropic part* of K.

Finally, in order to speak of continuity of a kernel as above, we need to assume that all the spheres involved are endowed with their geodesic distances. A nice discussion on continuous, isotropic and positive definite kernels on a single sphere, including applications, is made on the recent paper [8]. Additional information can be found in references therein. As for positive definiteness on a product of spheres, we have found no relevant references to quote, except [10].

For $m, M < \infty$, a result proved in [10] reveals that a real, continuous and isotropic kernel K on $S^m \times S^M$ is positive definite if, and only if, its isotropic part has a double Fourier series representation in the form

$$K_r(t,s) = \sum_{k,l=0}^{\infty} a_{k,l} P_k^m(t) P_l^M(s), \quad t,s \in [-1,1],$$

in which $a_{k,l} \geq 0$, $k,l \in \mathbb{Z}_+$, P_k^m is the Gegenbauer polynomial of degree k with respect to the real number $(m-1)/2$, and

$$\sum_{k,l=0}^{\infty} a_{k,l} P_k^m(1) P_l^M(1) < \infty.$$

Obviously, this theorem extends a famous result of I.J. Schoenberg [13] to products of spheres. The Gegenbauer polynomials appearing above are discussed in [7,14]. In particular, one may find there the orthogonality relation for them

$$\int_{-1}^{1} P_n^m(t) P_k^m(t)(1-t^2)^{(m-2)/2} dt = \frac{\tau_{m+1}}{\tau_m} \frac{m-1}{2n+m-1} P_n^m(1)\delta_{n,k},$$

in which τ_{m+1} is the surface area of S^m, that is,

$$\tau_{m+1} := \frac{2\pi^{(m+1)/2}}{\Gamma((m+1)/2)}.$$

If one or both spheres coincide with S^∞, the representation theorem described above still holds. Indeed, it suffices to replace each Gegenbauer polynomial with the standard monomial of equal degree in the appropriate spots in the expansion for K_r.

The results in this paper will apply to the case $m, M \geq 2$ only; in the other cases, some of the corresponding questions are still open while at least one has been settled already (see either comments ahead or [11]). Thus, throughout the paper, we will assume that $m, M \geq 2$. For a real, continuous, isotropic and positive definite kernel K on a product of spheres, we can define the set

$$J_K := \{(k,l) \in \mathbb{Z}^2 : a_{k,l} > 0\}$$

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