J. Math. Anal. Appl. 435 (2016) 315–320

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Lifting to the spectral ball with interpolation

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ARTICLE INFO

Article history:
Received 17 May 2015
Available online 23 October 2015
Submitted by N. Young

Keywords:
Spectral ball
Nevanlinna–Pick
Symmetrized polydisc
Oka principle
Interpolation

ABSTRACT

We give necessary and sufficient conditions for solving the spectral Nevanlinna–Pick lifting problem. This reduces the spectral Nevanlinna–Pick problem to a jet interpolation problem into the symmetrized polydisc.

1. Introduction

The spectral ball is the set of square matrices with spectral radius less than 1. It appears naturally in Control Theory [4,5], but is also of theoretical interest in Several Complex Variables.

Definition 1.1. The *spectral ball* of dimension $n \in \mathbb{N}$ is defined to be

$$\Omega_n := \{A \in \mathrm{Mat}\,(n \times n; \mathbb{C}) \ : \ \rho(A) < 1\}$$

where ρ denotes the spectral radius, i.e. the modulus of the largest eigenvalue.

Note that in dimension $n = 1$, the spectral ball is just the unit disc. We will assume throughout the paper that $n \geq 2$.

The Nevanlinna–Pick problem is an interpolation problem for holomorphic functions on the unit disc $\mathbb{D}$. The classical *Nevanlinna–Pick problem* for holomorphic functions $\mathbb{D} \to \mathbb{D}$ with interpolation in a finite set of points has been solved by Pick [13] and Nevanlinna [10]. The *spectral Nevanlinna–Pick problem* is the analogue interpolation problem for holomorphic maps $\mathbb{D} \to \Omega_n$:

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http://dx.doi.org/10.1016/j.jmaa.2015.10.020

Given $m \in \mathbb{N}$ distinct points $a_1, \dots, a_m \in \mathbb{D}$, decide whether there is a holomorphic map $F\colon \mathbb{D} \to \Omega_n$ such that

$$F(a_j) = A_j, \; j = 1, \dots, m$$

for given matrices $A_1, \dots, A_m \in \Omega_n$.

It has been studied by many authors, in particular by Agler and Young for dimension $n = 2$ and generic interpolation points [1,2]. Bercovici [3] has found solutions for $n = 2$, and Costara [6] has found solutions for generic interpolation points in higher dimensions. In general, the problem is still open.

The spectral ball Ω_n can also be understood in the following way: Denote by $\sigma_1, \dots, \sigma_n\colon \mathbb{C}^n \to \mathbb{C}$ the elementary symmetric polynomials in n complex variables. Let $\mathrm{EV}\colon \mathrm{Mat}\,(n \times n; \mathbb{C}) \to \mathbb{C}^n$ assign to each matrix a vector of its eigenvalues. Then we denote by $\pi_1 := \sigma_1 \circ \mathrm{EV}, \dots, \pi_n := \sigma_n \circ \mathrm{EV}$ the elementary symmetric polynomials in the eigenvalues. By symmetrizing we avoid any ambiguities of the order of eigenvalues and obtain a polynomial map $\pi = (\pi_1, \dots, \pi_n)$, symmetric in the entries of matrices in $\mathrm{Mat}\,(n \times n; \mathbb{C})$, actually

$$\chi_A(\lambda) = \lambda^n + \sum_{j=1}^{n} (-1)^j \cdot \pi_j(A) \cdot \lambda^{n-j}$$

where χ_A denotes the characteristic polynomial of A.

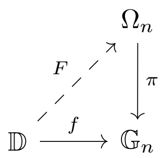
Now we can consider the holomorphic surjection $\pi\colon \Omega_n \to \mathbb{G}_n$ of the spectral ball onto the symmetrized polydisc $\mathbb{G}_n := (\sigma_1, \dots, \sigma_n)(\mathbb{D}^n)$. A generic fibre, i.e. a fibre above a base point with no multiple eigenvalues, consists exactly of one equivalence class of similar matrices. Thus, a generic fibre is actually a $\mathrm{SL}_n(\mathbb{C})$-homogeneous manifold where the group $\mathrm{SL}_n(\mathbb{C})$ acts by conjugation. A singular fibre decomposes into several strata which are $\mathrm{SL}_n(\mathbb{C})$-homogeneous manifolds as well, but not necessarily connected.

Given this holomorphic surjection, it is natural to consider a weaker version of the spectral Nevanlinna–Pick problem, which is called the *spectral Nevanlinna–Pick lifting problem*:

Given $m \in \mathbb{N}$ distinct points $a_1, \dots, a_m \in \mathbb{D}$, and a holomorphic map $f\colon \mathbb{D} \to \mathbb{G}_n$ with $f(a_j) = \pi(A_j)$ for given matrices $A_1, \dots, A_m \in \Omega_n$, decide whether there is a holomorphic map $F\colon \mathbb{D} \to \Omega_n$ such that

$$F(a_j) = A_j, \; j = 1, \dots, m$$

i.e. such that the following diagram commutes:



$$a_1, \dots, a_m \mapsto \pi(A_1), \dots, \pi(A_m)$$

When this lifting problem is solved, the spectral Nevanlinna–Pick problem reduces to an interpolation problem $\mathbb{D} \to \mathbb{G}_n$. In contrast to the spectral ball, the symmetrized polydisc $\mathbb{G}_n$ is a taut domain, and should be more accessible with techniques from hyperbolic geometry. Solutions to this lifting problem have been found for dimensions $n = 2, 3$ by Nikolov, Pflug and Thomas [11] and recently for dimensions $n = 4, 5$ by Nikolov, Thomas and Tran [12]. They also provide the solution to a localised version of the spectral Nevanlinna–Pick lifting problem.